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Fundamental Algebraic Geometry

Grothendieck's FGA Explained

Barbara Fantechi
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American Mathematical Society

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Preface

Without question, Alexander Grothendieck's work revolutionized Algebraic Geometry. He introduced many concepts — arbitrary schemes, representable functors, relative geometry, and so on — which have turned out to be astoundingly powerful and productive.

Grothendieck sketched his new theories in a series of talks at the Séminaire Bourbaki between 1957 and 1962, and collected his write-ups in a volume entitled "Fondements de la géométrie algébrique," commonly abbreviated FGA. In [**FGA**], he developed the following themes, which have become absolutely central:

- Descent theory,
- Hilbert schemes and Quot schemes,
- The formal existence theorem,
- The Picard scheme.

(FGA also includes a sketch of Grothendieck's extension of Serre duality for coherent sheaves; this theme is already elaborated in a fair number of works, and is not elaborated in the present book.)

Much of FGA is now common knowledge. Some of FGA is less well known, and few geometers are familiar with its full scope. Yet, its theories are fundamental ingredients in most of Algebraic Geometry.

Mudumbai S. Narasimhan conceived the idea of a summer school at the International Centre for Theoretical Physics (ICTP) in Trieste, Italy, to teach these theories. But this school was to be different from most ICTP summer schools. Most focus on current research: important new results are explained, but their proofs are sketched or skipped. This school was to teach the techniques: the proofs too had to be developed in sufficient detail.

Narasimhan's vision was realized July 7–18, 2003, as the "Advanced School in Basic Algebraic Geometry." Its scientific directors were Lother Göttsche of the ICTP, Conjeeveram S. Seshadri of the Chennai Mathematical Institute, India, and Angelo Vistoli of the Università di Bologna, Italy. The school offered the following courses:

(1) Angelo Vistoli: Grothendieck topologies and descent, 10 hours.
(2) Nitin Nitsure: Construction of Hilbert and Quot schemes, 6 hours.
(3) Lother Göttsche: Local properties of Hilbert schemes, and Hilbert schemes of points, 4 hours.
(4) Luc Illusie: Grothendieck's existence theorem in formal geometry, 5 hours.
(5) Steven L. Kleiman: The Picard scheme, 6 hours.

The school addressed advanced graduate students primarily and beginning researchers secondarily; both groups participated enthusiastically. The ICTP's administration was professional. Everyone had a memorable experience.

This book has five parts, which are expanded and corrected versions of notes handed out at the school. The book is not intended to replace [**FGA**]; indeed, nothing can ever replace a master's own words, and reading Grothendieck is always enlightening. Rather, this book fills in Grothendieck's outline. Furthermore, it introduces newer ideas whenever they promote understanding, and it draws connections to subsequent developments. For example, in the book, descent theory is written in the language of Grothendieck topologies, which Grothendieck introduced later. And the finiteness of the Hilbert scheme and of the Picard scheme, which are difficult basic results, are not proved using Chow coordinates, but using Castelnuovo–Mumford regularity, which is now a major tool in Algebraic Geometry and in Commutative Algebra.

This book is not meant to provide a quick and easy introduction. Rather, it contains demanding detailed treatments. Their reward is a far greater understanding of the material. The book's main prerequisite is a thorough acquaintance with basic scheme theory as developed in the textbook [**Har77**].

This book's contents are, in brief, as follows. Lengthier summaries are given in the introductions of the five parts.

Part 1 was written by Vistoli, and gives a fairly complete treatment of descent theory. Part 1 explains both the abstract aspects — fibered categories and stacks — and the most important concrete cases — descent of quasi-coherent sheaves and of schemes. Part 1 comprises Chapters 1–4.

Chapter 1 reviews some basic notions of category theory and of algebraic geometry. Chapter 2 introduces representable functors, Grothendieck topologies, and sheaves; these concepts are well known, and there are already several good treatments available, but the present treatment may be of greater appeal to a beginner, and can also serve as a warm-up to the more advanced theory that follows.

Chapter 3 is devoted to one basic notion, *fibered category*, which Grothendieck introduced in [**SGA1**]. The main example is the category of quasi-coherent sheaves over the category of schemes. Fibered categories provide the right abstract set-up for a discussion of descent theory. Although the general theory may be unnecessary for elementary applications, it is necessary for deeper comprehension and advanced applications.

Chapter 4 discusses *stacks*, fibered categories in which descent theory works. Chapter 4 treats, in full, the various ways of defining descent data, and it proves the main result of Part 1, which asserts that quasi-coherent sheaves form a stack.

Part 2 was written by Nitsure, and covers Grothendieck's construction of Hilbert schemes and Quot schemes, following his Bourbaki talk [**FGA**, 221], together with further developments by David Mumford and by Allen Altman and Kleiman. Part 2 comprises Chapter 5.

Specifically, given a scheme X, Grothendieck solved the basic problem of constructing another scheme Hilb_X, called the *Hilbert scheme of* X, which parameterizes, in a suitable universal manner, all possible closed subschemes of X. More generally, given a coherent sheaf E on X, he constructed a scheme $\mathrm{Quot}_{E/X}$, called the *Quot scheme of* E, which parameterizes, again in a suitable universal manner, all possible coherent quotients of E. These constructions are possible, in a relative set-up, where X is projective over a suitable base. The constructions make crucial use of several basic tools, including faithfully flat descent, flattening stratification, the semi-continuity complex, and Castelnuovo–Mumford regularity.

Part 3 was written jointly by Barbara Fantechi and Göttsche. It comprises Chapters 6 and 7.

Chapter 6 introduces the notion of an (infinitesimal) deformation functor, and gives several examples. Chapter 6 also defines a tangent-obstruction theory for such a functor, and explains how the theory yields an estimate on the dimension of the moduli space. The theory is worked out in some cases, and sketched in a few more, which are not needed in Chapter 7.

Chapter 7 studies the Hilbert scheme of points on a smooth quasi-projective variety, which parameterizes the finite subschemes of fixed length. The chapter constructs the Hilbert–Chow morphism, which maps this Hilbert scheme to the symmetric power by sending a subscheme to its support with multiplicities. For a surface, this morphism is a resolution of singularities. Finally, the chapter computes the Betti numbers of the Hilbert scheme, and sketches the action of the Heisenberg algebra on the cohomology.

Part 4 was written by Illusie, and revisits Grothendieck's Bourbaki talk [**FGA**, 182], where he presented a fundamental comparison theorem of "GAGA" type between algebraic geometry and formal geometry, and outlined some applications to the theory of the fundamental group and to that of infinitesimal deformations. A detailed account appeared shortly afterward in [**EGAIII1**], [**EGAIII2**] and [**SGA1**]. Part 4 comprises Chapter 8.

After recalling basic facts on locally Noetherian formal schemes, Chapter 8 explains the key points in the proof of the main comparison theorem, and sketches some corollaries, including Zariski's connectedness theorem and main theorem, and Grothendieck's criterion for algebraization of a formal scheme. Then Chapter 8 gives Grothendieck's applications to the fundamental group and to lifting vector bundles and smooth schemes, notably, curves and Abelian varieties. Chapter 8 ends with a discussion of Serre's celebrated examples of varieties in positive characteristic that do not lift to characteristic zero.

Part 5 was written by Kleiman, and develops in detail most of the theory of the Picard scheme that Grothendieck sketched in the two Bourbaki talks [**FGA**, 232, 236] and in his commentaries on them [**FGA**, pp. C-07–C-011]. In addition, Part 5 reviews in brief, in a series of scattered remarks, much of the rest of the theory developed by Grothendieck and by others. Part 5 comprises Chapter 9.

Chapter 9 begins with an extensive historical introduction, which serves to motivate Grothendieck's work on the Picard scheme by tracing the development of the ideas that led to it. The story is fascinating, and may be of independent interest.

Chapter 9 then discusses the four common relative Picard functors, which are successively more likely to be the functor of points of the Picard scheme. Next, Chapter 9 treats relative effective (Cartier) divisors and linear equivalence, two important preliminary notions. Then, Chapter 9 proves Grothendieck's main theorem about the Picard scheme: it exists for any projective and flat scheme whose geometric fibers are integral.

Chapter 9 next studies the union of the connected components of the identity elements of the fibers of the Picard scheme. Then, Chapter 9 proves two deeper finiteness theorems: the first concerns the set of points with a multiple in this union; the second concerns the set of points representing invertible sheaves with a given Hilbert polynomial. Chapter 9 closes with two appendices: one contains detailed answers to all the exercises; the other contains an elementary treatment of basic divisorial intersection theory — this theory is used freely in the proofs of the two finiteness theorems.

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Part 1

Grothendieck topologies, fibered categories and descent theory

Angelo Vistoli

Introduction

Descent theory has a somewhat formidable reputation among algebraic geometers. In fact, it simply says that under certain conditions homomorphisms between quasi-coherent sheaves can be constructed locally and then glued together if they satisfy a compatibility condition, while quasi-coherent sheaves themselves can be constructed locally and then glued together via isomorphisms that satisfy a cocycle condition.

Of course, if "locally" were to mean "locally in the Zariski topology" this would be a formal statement, certainly useful, but hardly deserving the name of a theory. The point is that "locally" here means locally in the flat topology; and the flat topology is something that is not a topology, but what is called a *Grothendieck topology*. Here the coverings are, essentially, flat surjective maps satisfying a finiteness condition. So there are many more coverings in this topology than in the Zariski topology, and the proof becomes highly nontrivial.

Still, the statement is very simple and natural, provided that one resorts to the usual abuse of identifying the pullback $(gf)^*F$ of a sheaf F along the composite of two maps f and g with f^*g^*F. If one wants to be fully rigorous, then one has to take into account the fact that $(gf)^*F$ and f^*g^*F are not identical, but there is a canonical isomorphism between them, satisfying some compatibility conditions, and has to develop a theory of such compatibilities. The resulting complications are, in my opinion, the origin of the distaste with which many algebraic geometers look at descent theory (when they look at all).

There is also an abstract notion of "category in which descent theory works"; the category of pairs consisting of a scheme and a quasi-coherent sheaf on it is an example. These categories are known as *stacks*. The general formalism is quite useful, even outside of moduli theory, where the theory of algebraic stacks has become absolutely central (see for example [**DM69**], [**Art74b**] and [**LMB00**]).

These notes were born to accompany my ten lectures on *Grothendieck topologies and descent theory* in the *Advanced School in Basic Algebraic Geometry* that took place at I.C.T.P., 7–18 July 2003. Their purpose is to provide an exposition of descent theory, more complete than the original (still very readable, and highly recommended) article of Grothendieck ([**Gro95b**]), or than [**SGA1**]. I also use the language of Grothendieck topologies, which is the natural one in this context, but had not been introduced at the time when the two standard sources were written.

The treatment here is slanted toward the general theory of fibered categories and stacks: so the algebraic geometer searching for immediate gratification will probably be frustrated. On the other hand, I find the general theory both interesting and applicable, and hope that at at least some of my readers will agree.

Also, in the discussion of descent theory for quasi-coherent sheaves and for schemes, which forms the real reason of being of these notes, I never use the convention of identifying objects when there is a canonical isomorphism between them, but I always specify the isomorphism, and write down explicitly the necessary compatibility conditions. This makes the treatment rigorous, but also rather heavy (for a particularly unpleasant example, see §4.3.3). One may question the wisdom of this choice; but I wanted to convince myself that a fully rigorous treatment was indeed possible. And the unhappy reader may be assured that this has cost more suffering to me than to her.

All of the ideas and the results contained in these notes are due to Grothendieck. There is nothing in here that is not, in some form, either in [**SGA1**] or in [**SGA4**], so I do not claim any originality at all.

There are modern developments of descent theory, particularly in category theory (see for example [**JT84**]) and in non-commutative algebra and non-commutative geometry ([**KR04a**] and [**KR04b**]). One of the most exciting ones, for topologists as well as algebraic geometers, is the idea of "higher descent", strictly linked with the important topic of higher category theory (see for example [**HS**] and [**Str03**]). We will not discuss any of these very interesting subjects.

Contents. In Chapter 1 I recall some basic notions in algebraic geometry and category theory.

The real action starts in Chapter 2. Here first I discuss Grothendieck's philosophy of representable functors, and give one of the main illustrative examples, by showing how this makes the notion of group scheme, and action of a group scheme on a scheme, very natural and easy. All of algebraic geometry can be systematically developed from this point of view, making it very clean and beautiful, and incomprehensible for the beginner (see [**DG70**]).

In Section 2.3 I define and discuss Grothendieck topologies and sheaves on them. I use the naive point of view of pretopologies, which I find much more intuitive. However, the more sophisticated point of view using sieves has advantages, so I try to have my cake and eat it too (the Italian expression, more vivid, is "have my barrel full and my wife drunk") by defining sieves and characterizing sheaves in terms of them, thus showing, implicitly, that the sheaves only depend on the topology and not on the pretopology. In this section I also introduce the four main topologies on the category of schemes, Zariski, étale, fppf and fpqc, and prove Grothendieck's theorem that a representable functor is a sheaf in all of them.

There are two possible formal setups for descent theory, fibered categories and pseudo-functors. The first one seems less cumbersome, so Chapter 3 is dedicated to the theory of fibered categories. However, here I also define pseudo-functors, and relate the two points of view, because several examples, for example quasi-coherent sheaves, are more naturally expressed in this language. I prove some important results (foremost is Yoneda's lemma for fibered categories), and conclude with a discussion of equivariant objects in a fibered category (I hope that some of the readers will find that this throws light on the rather complicated notion of equivariant sheaf).

The heart of these notes is Chapter 4. After a thorough discussion of descent data (I give several definitions of them, and prove their equivalence) I define the central concept, that of *stack*: a stack is a fibered category over a category with a Grothendieck topology, in which descent theory works (thus we see all the three notions appearing in the title in action). Then I proceed to proving the main theorem, stating that the fibered category of quasi-coherent sheaves is a stack in the fpqc topology. This is then applied to two of the main examples where descent theory for schemes works, that of affine morphisms, and morphisms endowed with a canonical ample line bundle. I also discuss a particularly interesting example, that of descent along principal bundles (torsors, in Grothendieck's terminology).

In the last section I give an example to show that étale descent does not always work for schemes, and end by mentioning that there is an extension of the concept of scheme, that of *algebraic space*, due to Michael Artin. Its usefulness is that on one hand algebraic spaces are, in a sense, very close to schemes, and one can define for them most of the concepts of scheme theory, and on the other hand fppf descent always works for them. It would have been a natural topic to include in the notes, but this would have further delayed their completion.

Prerequisites. I assume that the reader is acquainted with the language of schemes, at least at the level of Hartshorne's book ([**Har77**]). I use some concepts that are not contained in [**Har77**], such as that of a morphism locally of finite presentation; but I recall their main properties, with references to the appropriate parts of *Éléments de géométrie algébrique*, in Chapter 1.

I make heavy use of the categorical language: I assume that the reader is acquainted with the notions of category, functor and natural transformation, equivalence of categories. On the other hand, I do not use any advanced concepts, nor do I use any real results in category theory, with one exception: the reader should know that a fully faithful essentially surjective functor is an equivalence.

Acknowledgments. Teaching my course at the *Advanced School in Basic Algebraic Geometry* has been a very pleasant experience, thanks to the camaraderie of my fellow lecturers (Lothar Göttsche, Luc Illusie, Steve Kleiman and Nitin Nitsure) and the positive and enthusiastic attitude of the participants. I am also in debt with Lothar, Luc, Steve and Nitin because they never once complained about the delay with which these notes were being produced.

I am grateful to Steve Kleiman for useful discussions and suggestions, particularly involving the fpqc topology, and to Pino Rosolini, who, during several hikes on the Alps, tried to enlighten me on some categorical constructions.

I have had some interesting conversations with Behrang Noohi concerning the definition of a stack: I thank him warmly.

I learned about the counterexample in [**Ray70**, XII 3.2] from Andrew Kresch.

I also thank the many participants to the school who showed interest in my lecture series, and particularly those who pointed out mistakes in the first version of the notes. I am especially in debt with Zoran Skoda, who sent me several helpful comments, and also for his help with the bibliography.

Joachim Kock read carefully most of this, and send me a long list of comments and corrections, which were very useful. More corrections where provided by the referees. I am grateful to them.

Finally, I would like to dedicate these notes to the memory of my father-in-law, Amleto Rosolini, who passed away at the age of 86 as they were being completed. He would not have been interested in learning descent theory, but he was a kind and remarkable man, and his enthusiasm about mathematics, which lasted until his very last day, will always be an inspiration to me.

CHAPTER 1

Preliminary notions

1.1. Algebraic geometry

In this chapter we recall, without proof, some basic notions of scheme theory that are used in the notes. All rings and algebras will be commutative.

We will follow the terminology of *Éléments de géométrie algébrique*, with the customary exception of calling a "scheme" what is called there a "prescheme" (in *Éléments de géométrie algébrique*, a scheme is assumed to be separated).

We start with some finiteness conditions. Recall if B is an algebra over the ring A, we say that B is *finitely presented* if it is the quotient of a polynomial ring $A[x_1, \dots, x_n]$ over A by a finitely generated ideal. If A is noetherian, every finitely generated algebra is finitely presented.

If B is finitely presented over A, whenever we write $B = A[x_1, \dots, x_n]/I$, I is always finitely generated in $A[x_1, \dots, x_n]$ ([**EGAIV1**, Proposition 1.4.4]).

DEFINITION 1.1 (See [**EGAIV1**, 1.4.2]). A morphism of schemes $f\colon X \to Y$ is *locally of finite presentation* if for any $x \in X$ there are affine neighborhoods U of x in X and V of $f(x)$ in Y such that $f(U) \subseteq V$ and $\mathcal{O}(U)$ is finitely presented over $\mathcal{O}(V)$.

Clearly, if Y is locally noetherian, then f is locally of finite presentation if and only if it is locally of finite type.

PROPOSITION 1.2 ([**EGAIV1**, 1.4]).

(i) If $f\colon X \to Y$ is locally of finite presentation, U and V are open affine subsets of X and Y respectively, and $f(U) \subseteq V$, then $\mathcal{O}(U)$ is finitely presented over $\mathcal{O}(V)$.

(ii) The composite of morphisms locally of finite presentation is locally of finite presentation.

(iii) Given a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

if $X \to Y$ is locally of finite presentation, so is $X' \to Y'$.

DEFINITION 1.3 (See [**EGAI**, 6.6.1]). A morphism of schemes $X \to Y$ is *quasi-compact* if the inverse image in X of a quasi-compact open subset of Y is quasi-compact.

An affine scheme is quasi-compact, hence a scheme is quasi-compact if and only if it is the finite union of open affine subschemes; using this, it is easy to prove the following.