TOPOLOGICAL METHODS IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

MARSTON MORSE

TOPOLOGICAL METHODS IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

By

MARSTON MORSE
Institute for Advanced Study
Princeton, New Jersey

PRINCETON
PRINCETON UNIVERSITY PRESS
LONDON: HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
1947

Second printing 1951

Lithoprinted in U.S.A.
EDWARDS BROTHERS, INC.
ANN ARBOR, MICHIGAN
1951

FOREWORD

The following pages contain, in revised form, a set of lectures given at Fine Hall in Princeton, New Jersey, during the fall of 1945. A large part of the matter presented is the product of studies undertaken jointly by the author and Dr. Maurice Heins, reference to which is given in the bibliography. The first chapter on pseudo-harmonic functions is, however, derived largely from the author's paper on "The topology of pseudo-harmonic functions" Morse (1), while the fourth chapter on "The general order theorem" contains the first published proof of the theorem there stated. The present exposition differs from that in the joint papers, in that in the earlier papers attention was focused on meromorphic functions and the proofs then amended to include interior transformations. (See Stoilow (1), and Whyburn for previous work on interior transformations. With Whyburn our transformations are interior and "light".) In these lectures pseudo-harmonic functions and interior transformations are the starting point, and the theorems specialize into theorems on harmonic functions and meromorphic transformations.

The modern theory of meromorphic functions has distinguished itself by the fruitful use of the instruments of modern analysis and in particular by its use of the theories of integration. Its success along this line has perhaps diverted its attention from some of the more finitary and geometric aspects of function theory. Historically the geometric concepts of Riemann and Schwartz contrast with the more arithmetical concepts of Weierstrass and of the modern school*. The present lectures seek to emphasize again the advantages of geometric methods as a complement of other methods.

In the study of boundary values in a statistical sense, significant finite topological properties of the boundary images have been passed over, and the geometric instruments appropriate for simple generalization not always used. Passing to non-finitary aspects of the theory, the critical points of a harmonic function on a Jordan region, if infinite in number, stand in group theoretic or topological relation to the boundary values, assumed continuous, which arithmetic methods are not adequate to reveal. See Morse and Heins (1) III. On turning in still another direction of the theory, the topological development of pseudo-harmonic functions on the basis of the topological characteristics of their contour lines, makes the theory available, as Stefan Bergmann has pointed out, for the study of problems in partial differential equations not otherwise reached.

However, it is not these negative aspects which are most important but rather the possibility of attack on new problems of a fundamental nature. One of these problems is the determination of properties of deformation classes of meromorphic functions with prescribed zeros, poles and branch points. See Morse and Heins (2). Here a connection is made between the interest of the topologist in homotopy theories, and the classical interest in theorems on normal families, or covering theorems of the Picard type.

These lectures form merely the beginning of studies of this type. It is hoped that they may strike a responsive chord in the hearts of those to whom there is an appeal in the geometric approach.

*The remarkable work of Lars Ahlfors should be excepted.

TABLE OF CONTENTS

TABLE OF CONTENTS

CHAPTER I

PSEUDO-HARMONIC FUNCTIONS

§1. Introduction

We shall consider meromorphic functions $F(z)$ on a region G (open) bounded by v Jordan curves

$$(B_1, \ldots, B_v) = (B). \tag{1.1}$$

We shall suppose that $F(z)$ is defined on $\overline{G}$ (the closure of G), and is analytic on G except for poles, and continuous at points of (B). The number of poles of $F(z)$ on G is necessarily finite.

Alongside of $F(z)$ we shall consider <u>interior transformations</u> $w = f(z)$ of G into the w-sphere. Such transformations are generalizations of meromorphic functions. To define such a transformation one begins with a definition of an interior transformation in the neighborhood of an arbitrary point z_0 of G. Suppose that $F(t)$ is a non-constant, analytic function defined on a neighborhood N of t_0. One subjects N to a 1 - 1 continuous sense-preserving transformation

$$t = \phi(z) \qquad (t_0 = \phi(z_0)) \tag{1.2}$$

which maps N onto a neighborhood N_1 of z_0. The function

$$F(\phi(z)) = f(z) \tag{1.3}$$

thereby defined on N_1 is called an interior transformation $w = f(z)$ of N_1 into the w-sphere. A transformation $w = f(z)$ defined on G will be termed an interior transformation of G if $w = f(z)$ is an interior transformation of some neighborhood of each point of G.

We shall admit the possibility that $F(t)$ have a pole at t_0 and then say that $f(z)$ in (1.3) is an interior transformation with a pole at z_0. We shall consider interior transformations with at most a finite number of poles on G, and suppose that $f(z)$ is defined on $\bar{G}$ and continuous at points of (B). We do not say that $f(z)$ is an interior transformation on the boundary (B), although it is clear that $f(z)$ might in certain cases be extended in definition so as to be an interior transformation of a neighborhood of each boundary point.

We add an example of an interior transformation. Let $F(t)$ be an arbitrary polynomial in t. Set $z = x + iy$. Replace t in $F(t)$ by

$$t = 2x + iy = \phi(z).$$

The resulting function $F(\phi(z)) = f(z)$ will be interior but not analytic.

Interior transformations have been introduced at the very beginning not because they are our principal object of study but because they furnish a convenient medium for illustrating the new topological methods. The zeros, poles, and branch points of $f(z)$ are a fundamental source of study in the classical theory of functions. What are the relations between their numbers under given boundary conditions? To what extent do they determine the meromorphic function either with or without a knowledge of the boundary values? Theorems of this type have been given by Radó, Stoilow, Walsh, Backlund, Lucas and others. Possibly the simplest of these theorems is that of Lucas, as follows. If $P(z)$ is any polynomial in z, the zeros of

$P'(z)$ are found in any convex region which contains the zeros of $P(z)$. Many of the theorems of the above authors have their generalizations for interior transformations.

We have referred to branch points. It is necessary to give this term a meaning in the case of interior transformations. As is well known, a non-constant meromorphic function $f(z)$ if restricted to a sufficiently small neighborhood of a point z_0, takes on every value w in a sufficiently small neighborhood of $w_0 = f(z_0)$ an integral number of times m, w_0 alone excepted. If $m > 1$, the inverse of $f(z)$ is said to have a branch point of order $m - 1$ at the point w_0. With the neighborhood of z_0 restricted as above, $f(z)$ defines a meromorphic element Any interior transformation obtained from a meromorphic element by a homeomorphic change of independent variable will be called an interior element. The totality of function values w remains unaltered. The neighborhood of w_0 is covered the same number m of times by the interior element as by the defining meromorphic element. It is therefore appropriate to say that the interior element defines a branch point of order $m - 1$ at w_0 whenever $m > 1$. It is clear that this branch point order depends only on the given interior element and does not vary with the various meromorphic elements which may be used to define it. The orders of zeros or poles of an interior element are similarly defined as the orders of the zeros or poles of defining meromorphic elements.

Methods. The definition of an interior transformation is such that $f'(z)$ does not exist in general. The classical use of the Cauchy integral

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

to find the difference between the number of zeros and poles of $f(z)$ within C is thus unavailable, at least in any a priori sense. Branch points at ordinary points cannot be located in general as zeros of $f'(z)$. In the classical theory $f'(z)$ is either null or infinite within C, or defines a direction represented by arc $f'(z)$. Vector methods can then be used to locate the zeros of $f'(z)$, as in the case of one of the proofs of the fundamental theorem on algebra. These vector methods fail in the general theory, at least in the absence of some effective change of independent variable in the large. More important are positive advantages of topological methods. The classical treatment of boundary values by means of an integral in general ignores extremal properties of boundary values, such for example as the extremal values of $|f(z)|$. The images g_i under $w = f(z)$ of the boundary curves B_i, if locally simple, have important topological properties which more than compensate for the lack of derivatives. (A closed curve g is termed locally simple if it is the continuous and locally 1 - 1 image of a unit circle.)

In a final section we shall introduce a deformation theory of interior or meromorphic functions, considering one-parameter families of such functions

$$w = F(z, t) \qquad (0 \leq t \leq 1)$$

where for each t, $F(z, t)$ is an interior transformation defined on G, and such that the point w varies continuously on the "extended" w-plane with both z and t. Such a one-parameter family of interior transformations will be termed a deformation of $F(z, 0)$ into $F(z, 1)$. We admit deformations in which the zeros, poles and branch point antecedents are held fast, and put functions $f(z)$ which can be thus deformed into each other, into the same restricted deformation class. Deformations are also admitted in which the number but not the position of the zeros, poles, and branch point antecedents are held fast.

Topological invariants of the admissible deformations have been determined which characterize the deformation classes whether restricted or unrestricted. A question of great interest is whether the deformation classes defined by a use of meromorphic functions alone are identical with those defined when the more general interior transformations are used. Details will not be given. For proofs see Morse and Heins (2).

In general one seeks to distinguish those basic theorems on meromorphic functions which can be established for meromorphic functions but not for interior transformations. One such theorem is the Liouville theorem that a function which is analytic in the finite z-plane and bounded in absolute value, is constant. This is not true if stated for interior transformations. One can indeed map the finite z-plane homeomorphically on the interior of the circle $|w| < 1$ by the interior transformation $w = z/(1 + |z|)$; defined for every finite z. Clearly f(z) is not constant. On the other hand, we shall see that many theorems hold equally well for meromorphic functions and interior transformations.

§2. Pseudo-harmonic functions

The study of meromorphic functions leads naturally to harmonic functions. In a similar manner the study of interior transformations leads to functions which we shall call pseudo-harmonic and shall presently define.

We begin by considering the function

$$U(x, y) = \log |f(z)| \tag{2.0}$$

in case f(z) is meromorphic. As is well known this is the real part of log f(z) and is accordingly harmonic whenever the continuous branches of log f(z) are analytic. Thus U(x, y) is harmonic at every point $z = x + iy$ not a zero or pole of f(z). Let $z = a$ be a zero or pole of

$f(z)$. Then $f(z)$ admits a representation

$$f(z) = (z - a)^m A(z) \qquad (A(a) \neq 0)$$

where $A(z)$ is analytic at $z = a$. Neighboring $z - a$, U thus has the form

$$m \log |z - a| + \omega(x, y)$$

where $\omega(x, y)$ is harmonic. The function U has a <u>logarithmic pole</u> at $z = a$. More generally one considers harmonic functions of the form

$$k \log |z - a| + \omega(x, y) \qquad (k \neq 0)$$

where k is real but not necessarily an integer.

The <u>critical points</u> of U in (2.0) in the ordinary sense are the points at which $U_x = U_y = 0$. By virtue of the Cauchy-Riemann differential equations, when $f(z) \neq 0$ each such critical point is a zero of

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)},$$

and is thus a zero of $f'(z)$. Thus the zeros and poles of $f(z)$ are reflected by the logarithmic poles of $U(x,y)$ and the zeros of $f'(z)$ by the critical points of U.

Before coming to the definition of a pseudo-harmonic function, it will be helpful to give a description of the level arcs through a given point (x_0, y_0) of a non-constant harmonic function U. We are concerned with the locus

$$U(x, y) - U(x_0, y_0) = 0 \qquad (2.1)$$

neighboring (x_0, y_0). The harmonic function U is the real part of an analytic function $f(z)$. If $z_0 = x_0 + iy_0$, $f(z) - f(z_0)$ vanishes at z_0 and has the form

$$f(z) - f(z_0) = (z - z_0)^m A(z) \quad (A(z_0) \neq 0.) \tag{2.2}$$

We shall make a conformal transformation of a neighborhood of z_0 following which the desired level curves will appear as straight lines. This conformal transformation has the form

$$w = (z - z_0)A^{1/m}(z), \tag{2.3}$$

where any continuous single-valued branch of the m^{th} root may be used. The transformation (2.3) is locally 1 - 1 and conformal neighboring z_0, since at z_0

$$\frac{dw}{dz} = A^{1/m}(z_0) \neq 0.$$

In terms of the variable w,

$$f(z) - f(z_0) = w^m.$$

If $w = u + iv$ the required level lines are the level lines through the origin of

$$R(u + iv)^m \qquad (R = \text{Real part})$$

for example, if $m = 2$, the level lines of $u^2 - v^2$. If (r, θ) are polar coordinates in the w-plane

$$w^m = r^m(\cos m\theta + i \sin m\theta).$$

Thus by virtue of the transformation from (x, y) to (u, v) to (r, θ)

(2.4) $$U(x, y) - U(x_0, y_0) = r^m \cos m\theta.$$

In the (u, v) plane the required level lines are rays on which cos mθ = 0. There are 2m of these rays, each making an angle of $\frac{\pi}{m}$ with its successor. For example, if m = 1, the directions are $\frac{\pi}{2}, \frac{3\pi}{2}$. If m = 2 the directions are

$$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4},$$

that is, the lines of slope ±1. Since our transformation from the (x, y) plane to the (u, v)-plane was conformal, it follows that the level curves through (x_0, y_0) consist of m curves without singularity, each making an angle of $\frac{\pi}{m}$ at (x_0, y_0) with its successor. Another way of putting this result follows.

THEOREM* 2.1. Let (x_0, y_0) be a point at which U is harmonic. Suppose U is not constant. There exists an arbitrarily small neighborhood N of (x_0, y_0) whose closure is the homeomorph of a plane circular disc such that (x_0, y_0) corresponds to the center of the disc and the locus

(2.5) $$U(x, y) - U(x_0, y_0) = 0$$

corresponds to a set of 2m rays leading from the disc center and making successive sectors of central angle $\frac{\pi}{m}$. As a variable point crosses any one of these level lines (except at (x_0, y_0)) the difference (2.5) changes sign.

*Theorem 2.1 stated for pseudo-harmonic functions will be termed Theorem 2.1a.

The first statement of the theorem is an immediate consequence of the mapping of the (x, y)-plane into the (u, v)-plane as above. One chooses the disc $r \leq r_0$ in the (u, v)-plane with r_0 so small that the mapping of the (x, y)-plane into the (u, v)-plane is 1 - 1 and conformal for $r \leq r_0$. The second statement of the theorem follows from (2.4) and the fact that cos $m\theta$ changes sign with increasing θ whenever it vanishes.

With U non-constant, the smallest value of m in the theorem is 1, in which case there is but one non-singular level curve through (x_0, y_0). A particular consequence of the theorem is that U can never assume a relative maximum or minimum at a point (x_0, y_0) neighboring which it is harmonic. For one sees that $U(x, y) - U(x_0, y_0)$ is both positive and negative in every neighborhood of (x_0, y_0).

Definition of pseudo-harmonic functions. Let u(x, y) be a function which is harmonic and not identically constant in a neighborhood N of a point (x_0, y_0). Let the points of N be subjected to an arbitrary sense-preserving homeomorphism T in which N corresponds to another neighborhood N' of (x_0, y_0) and the point (x, y) on N corresponds to a point (x', y') on N'. It will be convenient to suppose that (x_0, y_0) corresponds to itself under T. Under T set

$$u(x, y) = U(x', y'). \tag{2.6}$$

The function U(x', y') will be termed pseudo-harmonic on N'. This definition will be extended to the case where u(x, y) has a logarithmic pole at (x_0, y_0). In this case

$$u(x, y) = k \log |z - z_0| + \omega(x, y) \qquad (k \neq 0)$$

where $\omega(x, y)$ is harmonic in a neighborhood of (x_0, y_0).

Under the above homeomorphism T, relation (2.6) defines what is termed a pseudo-harmonic function with logarithmic pole at (x_0, y_0). More generally, we shall admit functions U(x, y) which are pseudo-harmonic, except for logarithmic poles, in some neighborhood of every point of the region G and are continuous on the boundary of G.

With the above definition of a pseudo-harmonic function, it is clear that the level curves of a function U which is pseudo-harmonic in the neighborhood of a point (x_0, y_0) of G are such that Theorem 2.1a holds (i.e., Theorem 2.1 with "pseudo-harmonic" replacing "harmonic"). As a corollary it follows that a pseudo-harmonic function assumes a finite relative maximum or minimum at no point of G.

§3. <u>Critical points of</u> U <u>on</u> G.

Points of G at which U $<$ c will be said to be <u>below</u> c; those at which U $>$ c, <u>above</u> c. Let (x_0, y_0) be a point of G not a logarithmic pole and set

$$U(x_0, y_0) = c.$$

Refer to Theorem 2.1a. This theorem gives a canonical representation of the level arcs of U ending at (x_0, y_0). The neighborhood N of (x_0, y_0) of Theorem 2.1a will be termed <u>canonical</u>. Any one of the open, connected subsets of N bounded by two successive arcs at the level c and the intercepted arc of the boundary of N will be called a <u>sector</u> of N. There are m sectors of N below c, and m sectors above c. If m = 1 the point (x_0, y_0) will be termed <u>ordinary</u>, otherwise <u>critical</u>. When m $>$ 1 the number m - 1 will be called the <u>multiplicity</u> of the critical point (x_0, y_0) of G. For our purposes the essential topological characteristic of these critical points is the existence of two or more sectors of a canonical neighbor-