Robert D. Hackworth

MATHEMATICAL SYSTEMS
Finite and Infinite

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PREFACE

This book is a direct result of my belief that a textbook should be an important instructional device for the teaching of mathematics. Outwardly, most educators would concur, but in practice mathematics texts generally serve only as scanty outlines for class lectures or simply as workbooks to supply problem sets for practice on lecture materials. To suggest as I do that a mathematics text should actually be read and that a student should actually learn from it is, in fact, contradictory to the past experiences of almost every student in American schools. There are various reasons for this situation, but no justifications for its continuation. I have tried to write a text that will teach mathematics if it is read and studied.

In this regard, I have been guided by the statement attributed to the famous eighteenth-century mathematician Joseph Louis Lagrange that a mathematician has acquired an understanding of a concept only when it is possible for him to approach the first man he meets on the street and explain the concept to him so that he understands it thoroughly. There are two interesting ideas interwoven in Lagrange's statement. First, the mathematician-teacher must consciously trace his own understanding of a concept back to its most fundamental bases. Secondly, any explanation of a concept that is developed from such fundamental bases can be understood by almost anyone. I heartily subscribe to Lagrange's statement. I believe that the student can understand and appreciate mathematics only if he sees it in all its beauty, advantages, and disadvantages as the mathematician does. To expect the student to find pleasure in isolated facts or computational skills is to disregard those characteristics of mathematics which have made it the delight of scholars for many centuries.

In substance this is a basic text for mathematics, but its study easily leads to some of the deeper aspects of the subject. The material and presentation require only a limited knowledge of mathematics on the part of the student. Because it is devoted to teaching what mathematics is, the text is primarily concerned with the structure of a variety of mathematical systems. In mathematics as in architecture the foundation of a structure is of prime importance, and therefore the basic components of mathematical systems receive the most attention in the text. Logical conclusions that can be drawn for each mathematical system are usually only briefly developed, but this can easily lead classroom discussions to levels far beyond the scope of the text.

It has been said that a person achieves fluency in a foreign language only when he finds himself capable of framing problems and their solutions in that language. In my opinion this is also true of mathematics. The true test of a student's knowledge of mathematics must be his ability to frame and attack problems within the restrictions of some mathematical structure. For this reason mathematical proofs are found in every chapter of this text—a departure from the point of view of many authors who avoid proofs on the grounds that they are usually too difficult. On the contrary, I attack proofs from the beginning in the belief that such work is a prerequisite to grasping the very essence of mathematics. This is not intended to frighten the prospective student, for the first proofs encountered are extremely simple and the level of difficulty increases only as the skills are developed.

The text has been written with a spiraling effect. That is, each chapter depends upon those which precede it for background information and motivation in the expansion of topics already discussed. This fact restricts the use of the book to the sequence of chapters as written, but presents no great problem for the teacher who desires to proceed further, because any chapter to be skipped can be covered by a reading assignment. One word of caution is necessary. Students who are not accustomed to reading mathematics may find that Chapters 1, 2, and 3 are deceptively easy to read, but since they set the stage for all that follows, some special attention should be given to binary operations, postulates, and proofs for infinite sets.

Grateful acknowledgment is made to all those people who have assisted me in the writing and revising of this text. Perhaps my best and most severe critics have been the many students who have been involved in the class-testing of these materials each term since the fall of 1966. Without their appreciation and urging it is doubtful that I would have continued to the final manuscript. Without their criticism and questions I could not claim this to be the strong text it has become.

Many colleagues have left their mark on these pages, but special thanks are due to Joseph Gould, chairman of the mathematics department at Clearwater, who created an atmosphere conducive to good teaching and to the development of improved teaching materials; Mrs. Jean Newton, department of mathematics, who class-tested the material extensively and made invaluable suggestions for its improvement; John Schluep, State University of New York at Oswego, and Philip Emig, San Fernando Valley State College, California who carefully reviewed the manuscript in preliminary and final form and also offered excellent suggestions which have been incorporated into this text. A final word of appreciation goes to Peggy Park and Walter Bishop of Holt, Rinehart and Winston Inc., who have been patient and helpful in the nurturing of the preliminary manuscript into its final form.

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TO THE STUDENT

This text was written to provide the reader with a basic understanding of mathematics as it is today and as it may change in the future.

It is perhaps a surprising fact that although most students have completed at least ten years of mathematics before entering college, the question "What is mathematics?" is difficult to answer. So much of the student's time and effort is devoted to learning arithmetic that the basic qualities of mathematics frequently are not recognized nor understood.

Mathematics is not limited to the study of numbers and their relationships, nor is it merely a subject devoted to solving practical problems. Mathematics is more than a set of rules to be memorized and applied mechanically in the appropriate situations. However, in thus attempting to broaden the popular meaning of the term we are not denying that mathematics is indeed often concerned, used, or misused with such material.

Perhaps a better description of mathematics is that it is a way of thinking. To do mathematics is to think mathematically. The work of mathematics is done in the head, not with the hands. A machine can be built to add, multiply, subtract, and divide, but the machine itself performs no mathematics because it does not think.

There may be readers at this point who question their ability to think mathematically. It is not to be expected that such a skill can be achieved merely by stating it as an objective. Nor is there a magic wand whose touch confers mastery of mathematical secrets. This entire text is devoted to the task of providing the reader with the necessary knowledge and insight to think mathematically. Rather than doubting his present ability to think mathematically, the student should devote himself to understanding the material that follows. He may thus come to see that the basic qualities of mathematics are both surprisingly simple and interesting.

To understand the material of this text only a limited mathematical education is required. If the reader is proficient in arithmetic he will have the ability to handle any of the mathematics assumed here as "basic."

Of much greater importance to the reader's success with this text is a flexibility of mind that will assist rather than handicap him in approaching new or different ideas which may seem to contradict his own experiences. This point of view was well expressed by a wise college professor who told his students to remember everything they had ever learned about mathematics—then immediately forget it. Past knowledge may assist you in many ways, but do not let it interfere with your ability to evaluate new situations nor prejudice your thinking about situations you have already encountered.

Because the intent of this text is to give you a "feel" for mathematics rather than an ability to work problems, the study skills for this book are probably different from those you have used in the past. This text is definitely *not* a workbook of exercises to be done in some prescribed manner. The reader will encounter very few exercises of this type. This is a mathematics text that is intended to be used in much the same way as a history or economics text. It offers a rather complete discussion and explanation of the basic concepts involved in the study of each topic presented. Since the discussion is both basic and complete, the student is expected to read the material and develop an understanding of its content prior to any class discussion of the topic. The instructor can expect his students to acquire an understanding of the fundamental issues from the text; the student can achieve this knowledge by the careful reading and rereading of each chapter before such material is presented in class. The book is intended as a true "teacher at home" or self-study text, and should be used as such for its maximum effect.

Because the book is intended as a reading text, the exercises are mainly teaching exercises. As such, most of them are fairly easy because their purpose is to enhance the explanations rather than slow the learning progress. Do not skip or gloss over any exercise in your reading because it appears too easy. Questions such as $5 + 3 = ?$ are not intended to insult your intelligence. When such questions appear you are expected to look for their meaning in the development of some concept as well as answering the question.

Since the exercises are mainly intended as teaching devices rather than practice in computation, two types of questions are used. One type extends the development of concepts explained earlier. The second motivates the material that is to follow it and is usually not preceded by any explanation.

In summary, the student should proceed to the main body of the text with these two ideas: (1) The intent of this text is to give you a "feel" for mathematics that will provide an appreciation if not an understanding of the depth of the subject; (2) the text is not a workbook to be used only after an assignment is given, but is designed to teach many of the basic concepts of this course *prior* to any class discussion of the topics.