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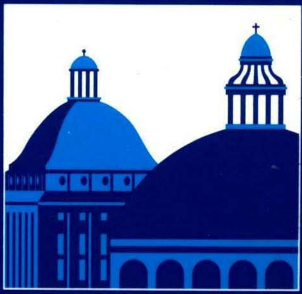

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European Mathematical Society

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0 Introduction

This short book is based on a lecture-course on the randomly forced two-dimensional Navier-Stokes Equation (2D NSE) and two-dimensional statistical hydrodynamics which the author taught at ETH-Zürich during the winter term of the year 2004/2005. The goal of the course was to review recent progress in the qualitative theory of randomly forced nonlinear PDE (especially, the 2D NSE), and discuss applications of the corresponding results to 2D statistical hydrodynamics, including 2D turbulence. The book, as well as the lecture-course, is aimed at people with some background in PDE, or in probability, or in physics. For the benefit of the last two groups of readers we included in the book a section on deterministic 2D NSE.

Due to the strictures of time, the lectures did not, and and this book does not, include all relevant material. The author restricts himself to results related to his current scientific interests – the statistical hydrodynamics of randomly forced two-dimensional fluids. Thus some important relevant topics are not represented in the book. Probably, the most serious omissions are results on the free NSE with random initial data. Concerning them we refer the reader to the books [VF88] and [FMRT01]. Some important results on randomly forced 2D fluids also are not covered by the book. With the exception of the very short Section 6.5 we avoid the randomly forced 3D NSE since not much is known about it, and what is known differs in spirit from the 2D results we are interested in. See [Fla05].

The book contains only rigorously proven theorems. Connections with the (heuristic) theory of turbulence are reduced to short discussions on relevance of the obtained results to the theory of turbulence, made at the ends of the main sections. There we show that the theorems form a rigorous mathematical foundation for the theory of 2D space-periodic turbulence. In particular, the results obtained imply that:

i) when time grows, statistical characteristics of a turbulent flow stabilize to characteristics independent of the initial velocity field (Sections 6, 7);
ii) for any characteristic of a turbulent flow, its time-average equals the ensemble-average (Section 8);
iii) in large time-scale the turbulent flow is a Gaussian process (Section 9).

In the last two sections of the book we prove and discuss some recent results, that seem to be unknown to experts in turbulence. Namely, we show that:

iv) when the coefficient of kinematic viscosity decays to zero and the random force, applied to the fluid, is scaled to keep the energy of the fluid of order one, the solution of the 2D NSE converges in distribution to a random field such that each of its realizations satisfies the free 2D Euler equation (Section 10);
v) stationary in space and time solutions of randomly forced 2D NSE satisfy infinitely many explicit algebraic relations (i.e., "space-periodic 2D turbulence is integrable"; Section 11).

The results i)–v) follow from rigorous analysis of the randomly forced 2D NSE

$$\dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \eta(t,x)\,, \quad \operatorname{div} u = 0\,, \tag{0.1}$$

where η is a random field. Usually the equation is supplemented by the periodic boundary conditions

$$x \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2\,.$$

Eq. (0.1) is the main object studied in this book. Sections 1–5 contain preliminaries, and the rest of the book treats new results on the equation (which imply the assertions i)–v) above).

Most of the results in Sections 6–9 hold true for eq. (0.1) in a bounded domain with suitable boundary conditions (say, Dirichlet), or in a two-dimensional compact Riemann surface, e.g., in a sphere (if the action of the Laplacian on vector-fields $u(x)$ is defined accordingly). More generally, the results hold for solutions of many nonlinear dissipative equations in bounded domains (or in a torus), perturbed by a random force. In particular, for the reaction-diffusion equation

$$\dot{u} - \nu\Delta u + u^3 = \eta(t,x)\,; \tag{0.2}$$

or for the Ginsburg-Landau equation

$$\dot{u} - \nu\Delta u + i|u|^2 u = \eta(t,x), \tag{0.3}$$

where $u(t,\,x) \in \mathbb{C}$ and $\dim x \le 3$; or for the equation

$$\dot{u} - (\nu + i)\Delta u + i|u|^2 u = \eta(t,x), \tag{0.4}$$

where $u(t,\,x) \in \mathbb{C}$, $\dim x \le 4$. From time to time we briefly discuss these equations and properties of their solutions, similar to those of 2D NSE.

In contrast, the results of Section 10 only hold for eq. (0.1) with *some* boundary conditions. For example, they do not apply to (0.1) with the Dirichlet boundary conditions, but they hold for the equation on a Riemann surface. Moreover, the results are valid for some other equations. In particular – for eq. (0.4).

The results of Section 11 are the most rigid: they only hold for the 2D NSE (0.1) under periodic boundary conditions (so only the periodic 2D turbulence is integrable, cf. v) above).

In this book we do not discuss properties of equations (0.2)–(0.4) which have no proven analogies for the 2D NSE (e.g., see [Kuk97, Kuk99] for a study of (0.3) when $\nu \to 0$). Similarly, we do not touch the problem of Burgers turbulence, described by the randomly forced Burgers equation (see [EKMS00]).

We consider two classes of random forces η: they are either Gaussian random fields, smooth in x and white as functions of t, or they are kick-processes as functions of t, smooth in x. In the former case the equations define stochastic (in Ito's sense) differential equations in function spaces, while in the latter case they define Markov chains in function spaces. All our results, apart from those in

Section 11, hold for both classes of forces. We think that this is important since it indicates that the results obtained for the 2D NSE (0.1) are not properties of a specific model, but of 2D statistical hydrodynamics.

Notation. We define $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$. For a Banach space X we set

$$\mathfrak{B}_r(X) = \{x \in X \mid \|x\|_X \leq r\}.$$

By $\mathcal{D}(\xi)$ we denote the distribution of a random variable ξ. Each metric space M is provided with the σ-algebra of its Borel sets $\mathcal{B}(\mathcal{M})$ (so 'measurable' means 'Borel-measurable'). We denote by $C_b(M)$ the space of bounded continuous functions on M, by $\mathcal{M}(M)$ – the set of finite signed Borel measures, and by $\mathcal{P}(M)$ – the probability Borel measures on M. For $f \in C_b(M)$ and $\mu \in \mathcal{M}(M)$ we define

$$(f, \mu) = (\mu, f) = \int_M f(u)\,\mu(du).$$

By I_Q we denote the characteristic function of a set Q.

We adopt the Einstein rule of summation over repeated indexes.

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Contents

1 Function spaces

1.1 Function spaces for functions of x

Let Q be an open domain of $\mathbb{R}^d$ or the torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$.

Lebesgue spaces. We denote by $L_p(Q;\mathbb{R}^n), 1 \le p \le \infty$, the usual Lebesgue spaces of vector-valued functions, abbreviated $L_p(Q;\mathbb{R}) = L_p(Q)$, and denote the L_2 scalar-product by $\langle \cdot, \cdot \rangle$.

Sobolev spaces $W^{m,p}(Q;\mathbb{R}^n)$. Let $C_c^\infty(Q;\mathbb{R}^n)$ be the space of infinitely differentiable maps $\phi : Q \to \mathbb{R}^n$ with compact support in Q. Suppose u, v are locally integrable functions on Q and $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written $D^\alpha u = v$, provided $\int_Q u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_Q v\phi \, dx$, for all test functions $\phi \in C_c^\infty(Q)$. Here $|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \cdots + \alpha_d$.

Let $m \in \mathbb{N}$ and $1 \le p \le \infty$. The space $W^{m,p}(Q, \mathbb{R}^n)$ consists of all locally integrable functions $u : Q \to \mathbb{R}^n$ such that for each multiindex α with $|\alpha| \le m$, $D^\alpha u$ exists in the weak sense and belongs to $L_p(Q;\mathbb{R}^n)$. We shall only use these spaces with $p = 2$ and adopt the notations:

$$W^{m,2}(Q;\mathbb{R}^n) = H^m(Q;\mathbb{R}^n), \quad H^m(Q;\mathbb{R}^1) = H^m(Q).$$

If $u \in H^m(Q;\mathbb{R}^n)$, we define its norm to be:

$$\|u\|_m = \Big(\sum_{|\alpha| \le m} |D^\alpha u|_2^2 \Big)^{1/2}.$$

When analysing eq. (0.1), we will mostly restrict ourselves to the case where the domain Q is the torus $\mathbb{T}^2 \stackrel{\text{def}}{=} \mathbb{R}^2/\mathbb{Z}^2$.

(S1) If $Q = \mathbb{T}^d$, then $u \in L_2$ can be written as $u(x) = \sum_{s\in\mathbb{Z}^d} u_s e^{is\cdot x}$. It is then possible to define $H^m(\mathbb{T}^d;\mathbb{R}^n)$ even for $m \in \mathbb{R}$. For this purpose, we define for any real number m a norm which is equivalent to the norm above if $m \in \mathbb{N}$:

$$\|u\|_m^2 = \sum_{s\in\mathbb{Z}^d} \big(1 + |s|^2\big)^m |u_s|^2, \quad m \in \mathbb{R}.$$

Now for $m \ge 0$ we set $H^m(\mathbb{T}^d;\mathbb{R}^n) = \{u \in L^2(\mathbb{T}^d;\mathbb{R}^n) : \|u\|_m^2 < \infty\}$, and for $m < 0$ we define the space $H^m(\mathbb{T}^d;\mathbb{R}^n)$ as the closure of $L_2(\mathbb{T}^d;\mathbb{R}^n)$ in the $\|\cdot\|_m$-norm.

Lemma 1.1. *For any $r \in \mathbb{R}$ and any multiindex α, the linear map D^α is continuous from $H^r(\mathbb{T}^d;\mathbb{R}^n)$ to $H^{r-|\alpha|}(\mathbb{T}^d;\mathbb{R}^n)$. Accordingly, the map $\Delta : H^r(\mathbb{T}^d;\mathbb{R}^n) \to H^{r-2}(\mathbb{T}^d;\mathbb{R}^n)$ is continuous.*

(S2) If $\langle u \rangle \stackrel{\text{def}}{=} \int_{\mathbb{T}^d} u(x)\, dx = 0$, then $u_0 = 0$. Therefore in the space

$$H_0^m(\mathbb{T}^d;\mathbb{R}^n) = \{u \in H^m(\mathbb{T}^d;\mathbb{R}^n) \mid \langle u \rangle = 0\},$$

the norm can be equivalently defined by the relation

$$\|u\|_m^2 = \sum_{s\neq 0} |s|^{2m} |u_s|^2 \,.$$

In particular, $\|u\|_1^2 = |\nabla u|_2$.

(S3) *Sobolev Embeddings.* Let Q be an open subset of $\mathbb{R}^d$ with a Lipschitz boundary, or the torus $\mathbb{T}^d$.

1. If $m \le \frac{d}{2}$ and $2 \le q \le \frac{2d}{d-2m}$, $q < \infty$, then
$$H^m(Q;\mathbb{R}^n) \subset L_q(Q;\mathbb{R}^n). \tag{1.1}$$

2. If $m > \frac{d}{2} + \alpha$, $0 \le \alpha < 1$, then
$$H^m(Q;\mathbb{R}^n) \subset C^\alpha(Q;\mathbb{R}^n). \tag{1.2}$$
$C^\alpha(Q)$, $\alpha > 0$, denotes the space of α-Hölder continuous functions and C^0 is the space of continuous functions.

3. If Q is an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, or if $Q = \mathbb{T}^d$, then the embedding (1.2) is compact, and the embedding (1.1) is compact as far as $q < \frac{2d}{d-2m}$. Besides, in this case
$$H^{m_1}(Q;\mathbb{R}^n) \Subset H^{m_2}(Q;\mathbb{R}^n) \quad \text{if } \ m_1 > m_2. \tag{1.3}$$

Exercise 1.2. *Prove* (1.2) *for* $\alpha = 0$ *and* $Q = \mathbb{T}^d$.

Solution: We have to show that $H^m(Q;\mathbb{R}^n) \subset C^0(Q;\mathbb{R}^n)$ if $m > \frac{d}{2}$. It is clear that $u = \sum_{s\in\mathbb{Z}^d} u_s e^{is\cdot x}$ with $\sum |u_s| < \infty$ is continuous. So it suffices to check that $\sum |u_s| < \infty$ for $u \in H^m(Q;\mathbb{R}^n)$ with $m > \frac{d}{2}$. We have:

$$\sum_{s\in\mathbb{Z}^d} |u_s|(1+|s|^2)^{-m/2}(1+|s|^2)^{m/2}$$
$$\le \left(\sum_{s\in\mathbb{Z}^d} |u|^2(1+|s|^2)^m\right)^{1/2} \left(\sum_{s\in\mathbb{Z}^d} (1+|s|^2)^{-m}\right)^{1/2} .$$

The first factor on the r.h.s. is finite since $u \in H^m(Q;\mathbb{R}^n)$. The second factor is finite because $2m > d$ and

$$\sum_{s\in\mathbb{Z}^d} (1+|s|^2)^{-m} \le 1 + \sum_{s\in\mathbb{Z}^d\setminus\{0\}} |s|^{-2m} \le c + c' \int_{|x|>1} |x|^{-2m} dx < \infty. \qquad \square$$

Example: If $Q = \mathbb{T}^2$, then

$$H^1(Q) \Subset L_q(Q) \quad \forall q < \infty, \tag{1.4}$$

and

$$H^{\frac{1}{2}}(Q) \subset L_4(Q). \tag{1.5}$$

(S4) The spaces $H^m(\mathbb{T}^d;\mathbb{R}^n)$ and $H^{-m}(\mathbb{T}^d;\mathbb{R}^n)$ are dual:

$$\forall u \in C^\infty(\mathbb{T}^d;\mathbb{R}^n), \quad \|u\|_m = \sup_{v\in C^\infty, \|v\|_{-m}\le 1} \langle u, v\rangle\,.$$

Exercise 1.3. *Prove this relation.*

In particular, the scalar product in L^2 extends to a continuous bilinear map $H^m(\mathbb{T}^d;\mathbb{R}^n) \times H^{-m}(\mathbb{T}^d;\mathbb{R}^n) \to \mathbb{R}$.

(S5) *Interpolation inequality.* Let $a < b$ and $0 \le \theta \le 1$. Then

$$\|u\|_{\theta a+(1-\theta)b} \le \|u\|_a^\theta \cdot \|u\|_b^{1-\theta}\,.$$

Proof (for the case $Q = \mathbb{T}^d$ and $\langle u\rangle = 0$). We have

$$\begin{aligned}
\|u\|^2_{\theta a+(1-\theta)b} &= \sum_{s\ne 0} |u_s|^2 |s|^{2(\theta a+(1-\theta)b)} \\
&= \sum_{s\ne 0} |u_s|^{2\theta} |s|^{2\theta a} |u_s|^{2(1-\theta)} |s|^{2(1-\theta)b} \\
&\le \left(\sum_{s\ne 0} |u_s|^2 |s|^{2a}\right)^{\theta} \left(\sum_{s\ne 0} |u_s||s|^{2b}\right)^{1-\theta},
\end{aligned}$$

where in the last step we use the Hölder inequality.

Example: In the example in (S3), using the interpolation with $a = 0, b = 1, \theta = \frac{1}{2}$, we get that $|u|_4 \le c\,\|u\|_{\frac{1}{2}} \le c\sqrt{|u|_2\,\|u\|_1}\,.$

1.2 Functions of t and x

The solutions of the equations, mentioned in the introduction, are functions depending on time t and space x. We fix $T > 0$ and view $u(t,x)$ with $0 \le t \le T$ as a map

$$[0,T] \longrightarrow \text{“space of functions of } x\text{”}, \qquad t \mapsto u(t,\cdot)\,.$$

Accordingly, we can define spaces

$$L_p\big(0,T;L_q(Q)\big) \stackrel{\text{def}}{=} L_p\big([0,T],L_q(Q)\big), \quad L_p\big(0,T;H^k(Q)\big) \stackrel{\text{def}}{=} L_p\big([0,T],H^k(Q)\big),$$
$$C\big(0,T;L_q(Q)\big) \stackrel{\text{def}}{=} C\big([0,T],L_q(Q)\big),$$

and so on. Fubini's theorem implies that

$$L_p\big(0,T;L_p(Q)\big) = L_p\big([0,T]\times Q\big)\,,$$

if $p < \infty$. Discussion of these spaces can be found in [Lio69] and [FMRT01].

We shall denote

$$C^\infty = \{u(t,x) \in C^\infty\} \text{ or } C^\infty = \{u(x) \in C^\infty\},$$

depending on the context.

Exercise 1.4. *Let $Q = \mathbb{R}^d$ and $T = 1$. Consider the heat kernel:*

$$u(t,x) = (2\sqrt{\pi t})^{-d} \exp\left(-\frac{|x|^2}{4t}\right).$$

Prove that

$$|u|_{L_p(0,1;L_2)} = \begin{cases} \infty, & pd \geq 4, \\ < \infty, & \text{otherwise.} \end{cases}$$

2 The deterministic 2D Navier-Stokes Equation

In the forthcoming, we write "2D NSE" for the "two-dimensional Navier-Stokes Equation", and often abbreviate 2D NSE to NSE. We will consider 2D NSE with periodic boundary conditions. That is, we assume that the space-variable x is a point in the torus $\mathbb{T}^2 \stackrel{\text{def}}{=} \mathbb{R}^2/2\pi\mathbb{Z}^2$. The Navier-Stokes Equation in a bounded two-dimensional domain under the Dirichlet boundary conditions can be studied in a very similar manner. In an unbounded domain, e.g., in the whole plane, the equation becomes somewhat complicated since in this case the Sobolev embeddings in (S3) are not compact.

The 2D NSE on the torus is the following system of three equations for three unknown functions:

- two components of the vector-function $u(t,x) = \left(u^1(t,x), u^2(t,x)\right)^t$ (the velocity) and
- the scalar function $p(t,x)$ (the pressure),

where $x \in \mathbb{T}^2$ and $t \in \mathbb{R}$:

$$\begin{cases} \dot{u}(t,x) - \nu\Delta u(t,x) + (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) = \tilde{f}(t,x), \\ \operatorname{div} u(t,x) = 0\,. \end{cases} \tag{2.1}$$

Usually we study the equation for $t \geq 0$ and supplement it with the initial condition at $t=0$:

$$u(\cdot,0) = u_0(\cdot)\,.$$

Standard references are, e.g., [Lio69, CF88], [BV92] and [FMRT01].

2.1 Leray decomposition

Let $u \in L_2(\mathbb{T}^2;\mathbb{R}^2)$, then u can be written as a Fourier series: $u(x) = \sum u_s e^{is\cdot x}$, with $u_s \in \mathbb{C}^2$ and $u_{-s} = \bar{u}_s$. If $u(x) \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$, then div $u(x) = \sum_{s\in\mathbb{Z}^2} is \cdot u_s e^{is\cdot x}$. Denote by H the space

$$H = \text{the closure in } L_2(\mathbb{T}^2;\mathbb{R}^2) \text{ of } \{u(x) \in C^\infty(\mathbb{T}^2;\mathbb{R}^2) |\ \operatorname{div} u = 0, \langle u\rangle = 0\}.$$

Then it holds that

$$H = \{u(x) = \sum_{s\in\mathbb{Z}_0^2} u_s e^{is\cdot x} \in L_2(\mathbb{T}^2;\mathbb{R}^2) \mid u_{-s} = \bar{u}_s,\ s\cdot u_s = 0\}\,, \tag{2.2}$$

where $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$. The norm in H will be denoted by $|\cdot|$, and the inner product by $\langle\cdot,\cdot\rangle$.

Exercise 2.1. *H can be defined as $H = \{u \in L_2 | \langle u\rangle = 0,\ div\ u = 0\}$, where the derivatives are viewed in the sense of generalised functions.*

We next introduce a basis of H. Let us define $\mathbb{Z}_+^2 = \{(s_1, s_2) \mid (s_1 > 0) \text{ or } (s_1 = 0, s_2 > 0)\}$. Then

$$\mathbb{Z}_0^2 = \mathbb{Z}_+^2 \cup (-\mathbb{Z}_+^2), \quad \mathbb{Z}_+^2 \cap -\mathbb{Z}_+^2 = \emptyset,$$

and we define the following set of vectors $\{e_s \mid s \in \mathbb{Z}_0^2\}$:

$$e_s = \begin{cases} c_s s^\perp \sin(s \cdot x), & s \in \mathbb{Z}_+^2, \\ c_s s^\perp \cos(s \cdot x), & s \in -\mathbb{Z}_+^2, \end{cases}$$

where $c_s = \frac{1}{\sqrt{2}\pi|s|}$, and if $s = (s_1, s_2)^t$, then $s^\perp = (-s_2, s_1)^t$. The set $\{e_s\}$ is a Hilbert basis of H, formed by eigenvectors of $-\Delta$:

$$-\Delta e_s = |s|^2 e_s \quad \forall s.$$

We further introduce the space

$$\nabla H^1 \stackrel{\text{def}}{=} \{\nabla f(x) | f \in H^1(\mathbb{T}^2)\}.$$

Equivalently,

$$\nabla H^1 = \{\sum_{s \in \mathbb{Z}_0^2} s a_s e^{is \cdot x} \in L_2(\mathbb{T}^2; \mathbb{R}^2)\}, \tag{2.3}$$

so ∇H^1 is a closed subspace of H.

The relations (2.2) and (2.3) immediately imply the following classical result due to Helmholtz, which since the works of Leray has become a common tool to study the Navier-Stokes equation:

Lemma 2.2. *The space $L_2(\mathbb{T}^2; \mathbb{R}^2)$ admits the following decomposition in a direct sum of three closed orthogonal subspaces*

$$L_2(\mathbb{T}^2; \mathbb{R}^2) = H \oplus \nabla H^1 \oplus \mathbb{R}^2,$$

where $\mathbb{R}^2$ stands for the space of constant vector-fields.

The orthogonal projection $\Pi : L_2(\mathbb{T}^2; \mathbb{R}^2) \to H$ is called the *Leray projection.* Note that

$$\Pi(\nabla p) = 0, \quad \Pi(constant) = 0.$$

Let $(u(t,x), p(t,x))$ be a smooth solution of (2.1). Let us denote $u(0,x) = u_0(x)$ and assume that

$$\langle u_0(x) \rangle = 0, \ \langle \tilde{f}(t,x) \rangle = 0 \text{ for all } t \geq 0.$$

Then, integrating the first equation of (2.1) over space, we obtain

$$\frac{d}{dt}\langle u \rangle - \langle \Delta u \rangle + \langle (u \cdot \nabla) u \rangle + \langle \nabla p \rangle = \langle \tilde{f} \rangle.$$

Since $\langle \Delta u \rangle = \langle \nabla p \rangle = \langle \tilde{f} \rangle = 0$ and for $l = 1, 2$ we have

$$\langle ((u \cdot \nabla)u)^l \rangle = \int u^j (\frac{\partial}{\partial x^j} u^l) dx = -\int \text{div } u \cdot u^l dx = 0 ,$$

then $\frac{d}{dt}\langle u \rangle = 0$. Hence, if $u_0 \in H$, then $u(t) \in H$ for all $t \geq 0$.

Now, apply the projection Π to (2.1). Since $\Pi u = u$, we find that

$$\dot{u} - \nu \Pi \Delta u + \Pi (u \cdot \nabla) u = \Pi \tilde{f} .$$

With the notation

$$Lu \overset{\text{def}}{=} -\Pi \Delta u = -\Delta u \text{ and } B(u) \overset{\text{def}}{=} \Pi (u \cdot \nabla) u, \text{ for } u \in H; \qquad f \overset{\text{def}}{=} \Pi \tilde{f} ,$$

we are led to the equation

$$\begin{cases} \dot{u} + \nu L u + B(u) = f(t) , \\ u(t) \in H . \end{cases} \tag{2.4}$$

Lemma 2.3. *If $u(t,x) \in C^\infty(\mathbb{T}^2)$ satisfies (2.4), then there exists $p(t,x) \in C^\infty(\mathbb{T}^2)$ such that (2.1) holds.*

Proof. Denote $\dot{u}(t,x) - \nu \Delta u(t,x) + B(u(t,x)) - \tilde{f}(t,x) = -\xi(t,x)$. Then $\Pi \xi = 0$. So, by Lemma 2.2, $\xi(t,x) = \nabla p(t,x) + C(t)$. For the same reasons as above, $C(t) \equiv 0$. □

Below we study eq. (2.4) instead of (2.1).

Notation: We set $B(u,v) = \Pi(u \cdot \nabla) v$ (so $B(u) = B(u,u)$). For $r \geq 0$ we define the space H^r as $H^r := H \cap H^r(\mathbb{T}^2, \mathbb{R}^2)$, and for $r < 0$ – as $H^r :=$ closure of H in $H^r(\mathbb{T}^2, \mathbb{R}^2)$. Then

$$H^r = \Big\{ u = \sum_{s \in \mathbb{Z}_0^2} u_s \, e_s(x) \mid \|u\|_r^2 = \sum |s|^{2r} |u_s|^2 < \infty \Big\} .$$

For $u \in H^r$ we have

$$\|u\|_r^2 = \langle L^r u, u \rangle ,$$

since $u \in H^r(\mathbb{T}^2, R^2)$ with $\langle u \rangle = 0$ satisfies $\|u\|_r^2 = \langle (-\Delta)^r u, u \rangle$ (see (S2)), and $L e_s = |s|^2 e_s$. In particular,

$$\|u\|_1^2 = \langle Lu, u \rangle = |\nabla u|_{L_2} , \quad \langle Lu, v \rangle = \langle \nabla u, \nabla v \rangle \tag{2.5}$$

for $u, v \in H^1$.

We note that $H^{r_1} \subset H^{r_2}$ if $r_1 \geq r_2$, that

$$\bigcap_r H^r = H \cap C^\infty$$

(this follows from (1.2)), and that the linear space $\cap_r H^r$ is dense in each space H^s.

Lemma 2.4 ("A bounded poly-linear map is continuous"). *If X_1, X_2 are Banach spaces and $F : X_1 \times \cdots \times X_1 \to X_2$ is a poly-linear map such that*

$$\|F(u_1, \ldots, u_r)\|_{X_2} \le C\|u_1\|_{X_1} \ldots \|u_r\|_{X_1},$$

then F is continuous. Moreover, if $V_1 \subset X_1$ is a dense linear subspace and the inequality above holds for $u_j \in V_1$, then F extends to a poly-linear continuous map $X_1 \times \cdots \times X_1 \to X_2$.

The proof of this result is straightforward.

2.2 Properties of the nonlinearity B

(B1) If u, v, $w \in C^\infty \cap H$, then

i) $\langle B(u,v), v\rangle = 0$,

ii) $\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle$.

Proof. i) Integrating by parts we have:

$$\begin{aligned}\langle B(u,v), v\rangle = \int_{\mathbb{T}^2} u^j \Big(\frac{\partial}{\partial x^j}\, v^l\Big)\, v^l\, dx &= \frac{1}{2} \int_{\mathbb{T}^2} u^j\, \frac{\partial}{\partial x^j}\, |v|^2\, dx \\ &= -\,\frac{1}{2} \int_{\mathbb{T}^2} (\operatorname{div} u)|u|^2 dx = 0\,.\end{aligned}$$

ii) Apply i) with $v := v + w$. □

(B2) If u, v, $w \in C^\infty \cap H$, then

i) $|\langle B(u,w), v\rangle| = |\langle B(u,v), w\rangle| \le C\|u\|_{1/2}\|v\|_{1/2}\|w\|_1$,

ii) $\|B(u,v)\|_{-1} \le C\|u\|_{1/2}\,\|v\|_{1/2}$.

Proof. i) implies ii) by the duality, see (S4). Now we prove i):

$$\begin{aligned}|\langle B(u,v), w\rangle| &\overset{\text{(B1)}}{=} |-\langle B(u,w), v\rangle| \le C \int |u|\;|\nabla w|\;|v| dx \\ &\overset{\text{Hölder}}{\le} C_1\, |\nabla w|_{L_2}\, |u|_{L_4}\, |v|_{L_4} \overset{(1.5)}{\le} C_2 \|w\|_1\, \|u\|_{1/2}\, \|v\|_{1/2}\,.\end{aligned}$$ □

So by Lemma 2.4 and (B2 ii), B extends to a bilinear continuous map,

$$B : H^{1/2} \times H^{1/2} \to H^{-1}\,.$$

(B3) If $u, v \in H \cap C^\infty$, then $\|B(u,v)\|_{-3} \le C|u|\,|v|$. So by Lemma 2.4, B extends to a continuous map $B : H \times H \to H^{-3}$.

The proof is left as an exercise.