THE NUMERICAL TREATMENT OF INTEGRAL EQUATIONS

By

CHRISTOPHER T. H. BAKER M.A., D.Phil.

THE NUMERICAL TREATMENT OF INTEGRAL EQUATIONS

By

CHRISTOPHER T. H. BAKER M.A., D.Phil.

Reader in Mathematics,
University of Manchester

CLARENDON PRESS · OXFORD
· 1977

Oxford University Press, Walton Street, Oxford OX2 6DP

OXFORD LONDON GLASGOW NEW YORK
TORONTO MELBOURNE WELLINGTON CAPE TOWN
IBADAN NAIROBI DAR ES SALAAM LUSAKA ADDIS ABABA
KUALA LUMPUR SINGAPORE JAKARTA HONG KONG TOKYO
DELHI BOMBAY CALCUTTA MADRAS KARACHI

ISBN 0 19 853406 X

Printed in Great Britain by
Thomson Litho Ltd., East Kilbride, Scotland.

PREFACE

... of making many books there is no end;
and much study is a weariness of the flesh. '
Ecclesiastes XI v. 12.

At the time of writing, few books are devoted to the numerical solution of integral equations. The book by Bückner (1952) is written in German, and that of Anselone (1971) is devoted principally to an abstract theory and its applications; the proceedings edited by Delves and Walsh (1974), which provides a useful theory, is, however, limited by space. Finally, the works of Atkinson (1976) and Ivanov (1976) will have been published by the time that this work appears; each relates to a restricted class of equation. It is therefore my hope that this book will go some way to fill what is, ostensibly, a gap in the English literature.

Hamming (1962) states that the purpose of computing is insight, not numbers. My own view can be similarly but more moderately expressed. In practice, any insight derived from computing follows from accurate (or controlled) computation. I assume, here, that the reader is interested in practical numerical methods for the approximate solution of integral equations, and insight into the extent of their accuracy and any limitations they may possess. By careful selection of their reading, those with varying requirements should find material on this subject to suit their individual tastes; I hope that this book will be helpful to those whose interest lies in the solution of a particular equation as well as those with wider mathematical interests.

In keeping with my general philosophy, I have isolated some of the more abstract theory, in Chapter 5. Unfortunately, this type of theory appears to be required to analyse the treatment of the more difficult problems which can occur in practice. The reader who prefers a succinct and fairly abstract treatment of such a theory may refer to Anselone (1971). Ivanov (1976) treats the signular equations mentioned in Chapter 5 in some detail.

The majority of the numerical methods discussed in this book are illustrated by simple test calculations, which were performed, in general, using the Atlas and ICL/CDC system provided for use of members of the University of Manchester.

The writing of this book has doubtless placed a strain upon the forebearance of both my colleagues and my family, and on the secretarial staff of the Department of Mathematics at Manchester and (whilst I was there on leave at the invitation of Professor Thos. E. Hull) of the Computer Science Department at the University of Toronto. In particular, the secretaries at Manchester have borne, ably, the brunt of typing the initial manuscript. My wife Helen assisted with the checking of the manuscript, proof-reading and helping in the preparation of the references and index. I wish to thank all those mentioned, and also all who have offered constructive criticism of the book, in particular (in alphabetical order) L. Fox, I. Gladwell, E.T. Goodwin, M.S. Keech, D. Kershaw, G.F. Miller, M.R. O'Donohoe, H.H. Robertson, A. Spence and K. Wright, all of whom have made a substantial contribution to relieving the burden on the author. To Professor L. Fox and Dr. E.T. Goodwin, in their rôles as editors of the series at the time of writing, go my thanks for their encouragement, whilst thanks also go to the staff of the Clarendon Press for their assistance.

Christopher T.H. Baker

Department of Mathematics
The Victoria University of Manchester
September 1975.

INTRODUCTION

This book is concerned with the numerical analysis of integral equations. We are not principally concerned with the abstract theory of integral equations, nor with applications of mathematics where integral equations arise, but the first chapter is devoted to a review of the theory of integral equations. The survey of certain aspects of numerical analysis in Chapter 2 is intended to emphasize various topics which are of relevance in the study of numerical methods for integral equations.

In practice, mathematical and physical insight is of value in any attempt to solve a particular equation. It would doubtless have proved instructive had I selected a particular equation and investigated the problem of approximating its solution, taking advantage of all that is known about the equation and its origin. Instead, I have chosen to write in general terms about various classes of equation.

I have generally tried to separate the exposition of numerical methods from the underlying theory of the methods. This has been done in order that the reader may pursue his own interests. However, practice not substantiated by theory is risky; equally, theory not motivated by practice is something of an affectation. The principal results of the theory are accessible in the statements of theorems. Theorems relating to convergence and order of convergence provide helpful insight into the behaviour of numerical methods, and may lead to rigorous error bounds or well-founded error estimates (both of which are desirable in practice as well as in theory).

Some theorems, of the type referred to above, may be viewed with a natural suspicion. In practical computation on a digital computer, limiting processes cannot take place and rounding errors occur which obscure the behaviour predicted by 'convergence theory'. A classic example occurs in the numerical solution of Volterra equations of the second kind, where a class of methods can be shown to be 'convergent' but not all are satisfactory in practice.

Whilst each of Chapters 3, 4, 5, and 6 is clearly divided into sections on practice and on theory, there are some areas where such a division cannot be properly maintained. This is most apparent in Chapter 5.

The book can also be divided into three parts, containing respectively Chapters 1 and 2 (which give background material), Chapters 3 and 4 (in which we consider certain types of *well-behaved* equations), and Chapters 5 and 6 (where the discussion includes coverage of some more difficult problems). Each chapter is prefaced by a summary of its contents, since it often happens that a reader consulting a book has a particular problem in mind.

Although a number of results and proofs which have not previously appeared in print are presented here, it has been my aim to give (within the constraints, and where possible) a discussion in terms of fairly fundamental concepts of analysis.

A word on the system of notation applied throughout the book is appropriate here. The length of the book and the range of topics covered do not permit a complete standardization of the notation, but certain principles are followed closely. Thus, real or complex scalars are denoted by *italic* or Greek symbols, as are functions assuming such values, whilst matrices are denoted by upper-case elite or Greek letters with wavy underlines (as in $\underset{\sim}{A}$ and $\underset{\sim}{\Sigma}$) and lower-case elite or Greek letters having wavy underlines, as in $\underset{\sim}{a}$ and $\underset{\sim}{\sigma}$, denote column vectors. Amongst functions, we reserve $f(x)$, if necessary with embellisments as in $f_r^{\pm}(x)$, to denote the solution of an integral equation. Computed approximations are denoted, in general, by adding a tilde as in $\tilde{f}(x)$; it is sometimes necessary to consider more than one type of approximation, however, and we then employ such symbols as $\hat{f}(x)$, $\mathfrak{f}(x)$, or $\tilde{\mathrm{f}}(x)$. Since the notation $\underset{\sim}{\mathrm{f}}$, $\underset{\sim}{\tilde{\mathrm{f}}}$, etc., is used to denote a vector whose components are values of the functions $f(x)$, $\tilde{f}(x)$, etc., consistency requires that functions denoted by $\tilde{f}(x)$ and by $\tilde{\mathrm{f}}(x)$ give rise to identical vectors $\underset{\sim}{\tilde{\mathrm{f}}}$.

These comments are intended to serve as an introduction to the system of notation to which we hope the reader will grow accustomed. It will be noted that certain Greek symbols serve to denote mathematical symbols (as Σ for summation) in addition to their normal rôles. Where I considered confusion possible, the symbol ε for membership is replaced by $\in$. Finally, it should be noted that the house style employed involves the repetition of an arithmetic operation at a break in a mathematical expression. The symbol * in the text concludes an Example.

CONTENTS

CONTENTS

1

INTEGRAL EQUATIONS

In Chapter 1 the reader is introduced to the theory of integral equations. We shall content ourselves with an outline of this theory, and a brief survey of some related mathematical material. This chapter is not intended to provide a complete theory, and in many cases results which are standard in the literature are quoted without proof. We shall cover in more detail various results which assume some importance in later parts of the book.

In this chapter, and elsewhere throughout the book, more space is devoted to non-singular linear integral equations than to non-linear or singular equations, although the latter equations may be more common in practice. We have three reasons for this: First, more is known about the linear theory; secondly, we can often gain some insight into the general case by considering simple examples; and thirdly, we can sometimes treat equations which are difficult to solve by considering related, but simpler, equations. Some indication of such techniques is given below; and I have taken pains to cover the types of equation which most commonly arise in practice.

It could be argued that the pedagogically correct technique for discussing integral equations is within an abstract framework of functional analysis. I have foresaken such an approach in the interest of simplicity, but have devoted sections to an introduction to the type of analysis used in the abstract discussion of integral equations.

The reader is asked to be selective in his study. A core of the material necessary for a discussion of numerical methods is given in sections 1.1-1.4 and 1.10-1.12, and the remainder of the chapter could be treated as a reference section for later use.

Should an extensive, systematic or rigorous development of the theory of integral equations be required, I would refer the reader (in particular) to the work of Smithies (1962), of Cochran (1972), and of Zabreyko, Koshelev, Krasnosel'skii, Mikhlin, Rakovshchik and Stet'senko (1975).

1.1. *General remarks*

A functional equation in which the unknown function appears under an integral sign is called an integral equation; see, however, Zabreyko *et al.* (1975, p.1). For example, the quations

$$\frac{1}{f_0(x)} + \tfrac{1}{2}\int_0^1 \frac{x f_0(y)}{x+y}\,dy = 1 \qquad (0 \le x \le 1)\,, \tag{1.1}$$

$$f_1(x) - \int_0^x (x-y)\,\{f_1(y)\}^2\,dy = e^{-x} \qquad (x \ge 0)\,, \tag{1.2}$$

$$\int_{-\infty}^{\infty} \exp\{-(x-y)^2\}\, f_2(y)\,dy = e^{x} \qquad (-\infty < x < \infty)\,, \tag{1.3}$$

and

$$\lambda \int_{-1}^{1}\int_{-1}^{1} |xu + yv|\, f_3(u,v)\,du\,dv = f_3(x,y) \quad (-1 < x,y < 1) \tag{1.4}$$

are all integral equations. The functions to be found are $f_0(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x,y)$ respectively. If the derivative of the solution appears in the integral equation, as in

$$\{f''(x)\}^2 - \int_{-1}^{1} \sin xy\, f(y)\,dy = \cos x \quad (-1 < x < 1)\,, \tag{1.5}$$

the equation is usually known as an integro-differential equation, and additional boundary conditions are required to determine the solution $f(x)$.

Integral equations occur in the mathematical theory for a number of branches of science. In particular, they occur in the study of acoustics, optics and laser theory, potential theory, radiative transfer theory, cardiology, and in fluid mechanics and statistics.

The solutions occur non-linearly in eqns (1.1), (1.2), and (1.5), and these equations are therefore said to be non-linear equations. The other examples are linear equations (the unknown function appears linearly). In the early parts of this book we treat linear equations (in which the solution is a function of a single variable, as in eqn (1.3)). Later in the book we shall consider more general types of integral equation.

One of our first tasks is to classify the different types of linear integral equation.

1.2. *Preliminary classification of linear integral equations*

A general form of linear integral equation is given by the equation

$$\alpha(x)\ f(x) - \lambda \int_a^b K(x,y)\ f(y)\mathrm{d}y = g(x)\ , \tag{1.6}$$

where the functions involved may be supposed to be complex-valued functions of real variables. We shall suppose that eqn (1.6) is valid for $a \leq x \leq b$, and we shall suppose, in general, that a and b are finite. The functions $\alpha(x)$, $g(x)$, and $K(x,y)$ are known for $a \leq x,\ y \leq b$, and λ is a constant (which, when its value is known, is sometimes absorbed in $K(x,y)$). The function $K(x,y)$ is called the *kernel* of the integral equation.

We shall consider some particular cases of (1.6). The first of these is

$$\int_a^b K(x,y)\ f(y)\mathrm{d}y = g(x)\ , \qquad (c \leq x \leq d)\ , \tag{1.7}$$

which is called an *equation of the first kind*. We can ensure, by a change of variable, that $c=a$ and $d=b$, given that $|abcd|$ is finite.

Example 1.1

The equation

$$\int_0^1 (x^2 + y^2)^{\frac{1}{2}} f(y)\mathrm{d}y = \frac{1}{3}\{(1 + x^2)^{\frac{3}{2}} - x^3\}\ (0 \leq x \leq 1)$$

has a solution $f(x) = x$. *

The equation

$$f(x) - \lambda\int_a^b K(x,y)\, f(y)\mathrm{d}y = g(x) \qquad (a \le x \le b) , \tag{1.8}$$

where λ is a *known* constant, is called an *equation of the second kind.*

Example 1.2

The equation

$$f(x) - \int_0^1 \mathrm{e}^{xy}\, f(y)\mathrm{d}y = 1 - \frac{1}{x}(\mathrm{e}^x - 1) \quad (0 \le x \le 1)$$

has the unique solution $f(x) = 1$. *

Example 1.3

The equation

$$f(x) - 3\int_0^1 xy\, f(y)\mathrm{d}y = x^2 \quad . \quad (0 \le x \le 1)$$

has no solution. *

Connected with eqn (1.8) is an *eigenvalue problem* associated with the equation

$$f(x) = \lambda\int_a^b K(x,y)\, f(y)\mathrm{d}y \qquad (a \le x \le b) \tag{1.9}$$

and eqn (1.10) below. In solving (1.9), we seek values of the parameter λ for which (1.9) has a non-null solution $f(x)$. Such a value λ is called a *characteristic value.*† Since $f(x)$ is non-null it is clear that $\lambda \neq 0$ and we may write $\kappa = 1/\lambda$, so that eqn (1.9) becomes

† Not all authors use our terminology. See, for example, Tricomi (1957).