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ESSENTIAL ALGEBRA

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ESSENTIAL ALGEBRA

Preface

The purpose of this text is to provide the beginning and intermediate algebra essential for the study of mathematics at the college level. It is designed for students with no knowledge of mathematics other than elementary computational arithmetic. Although the text was developed to be used by students attending a class taught by an instructor, the text can be used without classroom instruction. Full instructions are provided and an abundance of illustrative examples are included.

This text, second in a series, presents the algebra needed by those planning to take technical mathematics, mathematics for elementary school teachers, or other courses in collegiate mathematics such as elementary functions, trigonometry, finite mathematics, linear algebra, or calculus for business and social sciences. The contents include essential set terminology together with beginning and intermediate algebra, logarithms, and an introduction to functions and graphs. Chapters 1 and 2 consist mainly of excerpts from Essential Mathematics (Scott, Foresman and Company, 1972) by this author. Students who have completed Sections 1-1 through 1-4, Chapters 2 through 5, and Sections 6-2 and 6-5 of that text should begin with Chapter 3 of Essential Algebra.

A **pretest,** a detailed programmed **review,** and **two chapter tests** are included for **each chapter,** and a set of **exercises** is included for **each section.** The questions on the pretests are labeled by sections so that the pretests can conveniently be used for diagnostic purposes. The items on the programmed reviews are also labeled by sections so that the student can easily refer back to the text.

The tests, exercises, and programmed reviews may be used in a variety of ways. One way to use them is as follows: (1) pretest; (2) cover each section needed, assign and correct exercises on each such section; (3) assign programmed review; (4) administer one of the chapter tests; (5) if necessary, repeat review, and retest with the other chapter test. Since the tests and exercises may be removed without removing instructional material, they may be collected and corrected outside of class. Answers to odd-numbered exercises are included in the text, and answers to all tests and even-numbered exercises are available to the instructor.

Appendixes contain essential geometric formulas, tables of units of measure, a table of powers, and a table of common logarithms. An index is included.

In order to present the essentials of algebra together with various pedagogical aids in one volume, the author had to forego lengthy motivational discussions and extensive use of the discovery technique. Wherever it seemed feasible and space permitted, however, justifications for rules and methods have been given.

The bulk of this text is incorporated in a third volume of this series, Essential Algebra with Functions (Scott, Foresman and Company, 1973) also by this author. The contents of the third volume consist of most of the contents of Essential Algebra with the addition of trigonometric functions and other common elementary functions.

It is a pleasure to thank my sons, Fred and Tom, and my husband, Professor Fred D. Stockton, for their continued encouragement and understanding. It is also a pleasure to thank Theodore Djaferis and Joseph Will for preparing answers to tests and exercises and to thank C. Joseph Frank of Scott, Foresman and Company for ably and amicably directing publication of this project.

Doris S. Stockton

Amherst, Massachusetts

To the Student

This book contains ten chapters each divided into sections. There is a set of **exercises** at the end of **each section.** At the end of **each chapter** there is a programmed **review** of the chapter and two **chapter tests.**

An Appendix contains a **pretest** for each chapter. The questions in the pretests are labeled by sections so that the pretests may be used to determine the sections you should study.

Answers to odd-numbered exercises and an index are at the back of the book.

One way to use each chapter of this book is as follows:

1. Take the pretest.
2. Study each section indicated by results of pretest.
3. Do selections from the exercise for each section studied. Write each answer in the space provided. (In some instances, where space is limited, you may need to do your work on a separate work sheet to be handed in with the exercise sheet.)
4. Do the programmed review.
5. Take one of the chapter tests.
6. If necessary, repeat the programmed review and take the other chapter test.

Instructions for Programmed Reviews

1. Cover everything on the page below the first item (below the first heavy dot in the left margin) with a mask such as a piece of paper.
2. Respond by filling in the blank or blanks.
3. Move the mask down to uncover the given response.
4. Compare responses; if necessary, correct.
5. Go on to the next item.

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Chapter 1

Sets and Numbers

A familiarity with the vocabulary of sets and with the most commonly used sets of numbers is essential for anyone studying contemporary mathematics.

1–1 SETS

A **set** is a collection of things. The things in a set are called its **elements** or its **members.** Two examples of sets are:

The countries of North America,
The first four letters of the English alphabet.

It is customary to use capital letters to represent sets and to use the symbol = for "represents", "equals", "is equal to", "is identical with", or "is indistinguishable from". The symbol $\in$ means "is an element of" or "is a member of". The symbol $\notin$ means "is not an element of" or "is not a member of". Thus if

C = The countries of North America

and

F = The first four letters of the English alphabet,

we may write:

Canada $\in C$,
$\text{d} \in F$,
$\text{e} \notin F$.

One way to denote a set that just has a few elements is to list the elements and enclose the list in **braces:** { }.

$F = \{\text{a, b, c, d}\}$

The order of listing the elements is immaterial so that the set {a, b, c, d} may also be written {b, a, d, c} or {d, c, a, b}, etc.

If a set has more than just a few elements, it may be more convenient to use what is known as **set-builder notation.** As an illustration, the set C above may be denoted by

$$\{x \mid x \text{ is a country of North America}\}.$$

This is read "the set of all x such that x is a country of North America".

Regardless of the way in which a set is designated, it must be possible to determine exactly what the elements of a set are. For example,

$$\{x \mid x \text{ is an Iowa town with population less than } 1200\}$$

is a clearly defined set. However,

$$\{x \mid x \text{ is a small Iowa town}\}$$

is not clearly defined because the meaning of "small" has not been specified.

Some sets may be clearly designated by listing only a few of their members. For example, the set of all the letters of the English alphabet may be denoted by

$$\{\text{a, b, c, . . . , z}\}.$$

The three dots indicate that the letters d through y inclusive are also members of the set even though they have not been explicitly listed.

A set, such as the set of all the grains of sand on Daytona Beach, may have a very large number of elements. On the other hand, a set may have exactly one element. If we let

$$S = \text{The countries of North America containing the Statue of Liberty},$$

then

$$S = \{\text{United States}\}.$$

A set may even have exactly no elements. If

$$D = \text{The set of living dinosaurs},$$

then D has no elements because dinosaurs are extinct. Any set with no elements is said to be empty and is called the **null set** or the **empty set**. The empty set is commonly denoted by the symbol $\emptyset$.

Two sets are said to be *equal* provided that they have exactly the same elements.

Definition Set A is **equal** to set B, $A = B$, provided that every element of A is an element of B, and every element of B is an element of A.

The symbol $\neq$ is used for "is not equal to". Thus

$$\{a, b, c\} \neq \{a, b, z\}.$$

Definition A set A is a **subset** of a set B provided that every element of A is also an element of B.

The symbol $\subseteq$ is commonly used for "is a subset of".

$$\{a, b, c\} \subseteq \{a, b, c, d\}$$

When we write $A \subseteq B$, we include the possibility that A might be equal to B. The set B is actually a subset of itself because every element of B is an element of B.

$$B \subseteq B$$

The empty set $\emptyset$ is defined to be a subset of every set. For any set B we may write

$$\emptyset \subseteq B.$$

Example 1 List all the subsets of $\{a, b, c\}$.

Solution $\{a, b, c\}$, $\emptyset$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a\}$, $\{b\}$, $\{c\}$.

In any discussion involving sets, it is frequently convenient to have it understood from the beginning that the elements of all the sets involved belong to a previously specified set known as the **universal set**. For example, in discussing sets of people, it might be convenient to let the universal set be specified as the set of all people. In that case,

$$\{x \mid x \text{ is a person living in Maine}\}$$

could be more simply written as

$$\{x \mid x \text{ lives in Maine}\}.$$

With the universal set specified as the set of all people, x is *understood* to be a person. Note that with a different specification for the universal set, the notation $\{x \mid x \text{ lives in Maine}\}$ may denote an entirely different set. For example, if the universal set were the set of all deer, $\{x \mid x \text{ lives in Maine}\}$ would then denote the deer population in Maine. The choice of universal set depends upon the nature of the sets involved.

The symbol U is used for the universal set. Since every set is a subset of its universal set, for any set A we may write

$$A \subseteq U.$$

Definition For any subset A of a universal set U, the set of all elements of U which are *not* elements of A is called the **complement of A** and is denoted by A'.

For example, if

$$U = \{a, e, i, o, u\} \quad \text{and} \quad A = \{a, e, o\},$$

then

$$A' = \{i, u\}.$$

Some sets have elements in common. The set $A = \{a, b, c, d\}$ and the set $B = \{c, d, e, f\}$ have the elements c and d in common.

Definition The set of all elements common to sets A and B is called the **intersection** of A and B and is denoted by $A \cap B$.

In other words, the intersection of A and B is the set of all elements which are in A and also in B.

$$\{a, b, c, d\} \cap \{c, d, e, f\} = \{c, d\}$$

If two sets have no elements in common, their intersection is the empty set.

$$\{a, b, c\} \cap \{x, y, z\} = \emptyset$$

Sometimes it is necessary to discuss the set of all elements which are in one or in both of two sets.

Definition The **union** of any two sets A and B is the set of all elements which are in A or in B or in both A and B. The union of A and B is denoted by $A \cup B$.

$$\{a, b, c, d, e\} \cup \{b, c, d, f, g, h\} = \{a, b, c, d, e, f, g, h\}$$

The elements a and e are in this union because they are in $\{a, b, c, d, e\}$, the elements f, g, and h are in this union because they are in $\{b, c, d, f, g, h\}$, and the elements b, c, d are in this union because they are in both $\{a, b, c, d, e\}$ and $\{b, c, d, f, g, h\}$. Those elements which happen to be common to the two sets need not be listed twice in the union.

Example 2 If $A = \{5, 6, 7, 8, 9\}$ and $B = \{4, 5, 6, 7\}$, find $A \cap B$ and $A \cup B$.

Solution

$$A \cap B = \{5, 6, 7\}$$
$$A \cup B = \{5, 6, 7, 8, 9, 4\}$$

Name ______________________ Date ______________________

EXERCISE 1–1 SETS

1. List the elements of the set of all days of the week whose names begin with the letter T. Use braces to denote the set. ____________

2. List the elements of the set of all months of the year whose names end with the letter Y. Use braces to denote the set. ____________

3. Use braces to denote the set of letters in the word college. ____________

4. Use braces to denote the set of face cards in a standard deck of playing cards. ____________

5. Passenger pigeons are extinct. The last one died in the Cincinnati zoo in 1914. What is the set of living passenger pigeons? ____________

6. What is the set of all people with three biological parents? ____________

7. Use set-builder notation to denote the set of all cab drivers. ____________

8. Use set-builder notation to denote the set of all people who live in the Bronx. ____________

9. Use set-builder notation to denote the set of all felines. ____________

10. Use set-builder notation to denote the set of all people listed in the Manhattan telephone directory. ____________

11. List the elements of $\{x \mid x$ is one of the first eight letters of the English alphabet$\}$. ____________

12. List the elements of $\{x \mid x$ is a letter in the word girl$\}$. ____________

13. Is $\{r, a, t\} = \{t, a, r\}$? Justify your answer. ____________

14. Is $\{7, 8\} = \{8, 7\}$? Justify your answer. ____________

15. Is $\{1, 2\} = \{2, 4\}$? Justify your answer. ____________

16. Is $\{2, 3\} = \{4, 6\}$? Justify your answer. ____________

17. Is $\{x \mid x$ is a weekday whose name begins with M$\}$ equal to {Monday}? Justify your answer. ____________

18. Is $\{x \mid x$ is a season of the year whose name begins with w$\}$ equal to {winter}? Justify your answer. ____________

19. How many elements are there in the set $\{\emptyset\}$? How many elements are there in $\emptyset$? Is $\{\emptyset\} = \emptyset$? ____________

20. List all the subsets of $\{3, 4\}$. ____________

21. List all the subsets of $\{x, y, z\}$. ____________

22. List all the subsets of $\{x, y, z, w\}$. ____________

23. List all the subsets of $\{a, b, c, d\}$. ____________

24. Is it true that $\{r, s, t\} \subseteq \{r, s, t\}$? Justify your answer. ____________

25. Is it true that $\{u, v, w\} \subseteq \{v, u, w\}$? ____________

26. Is it true that $\emptyset \subseteq \{a, b\}$? ____________

27. Is $\{x \mid x$ is a person who can live without water$\}$ a subset of $\{u, v, w\}$? ____________

28. Make an appropriate choice for the universal set, if the sets under discussion are:
P = {teenager, construction worker, housewife}
Q = {surfer, skier}. ____________

29. Make an appropriate choice for the universal set, if the sets under discussion are:
P = {golfer, swimmer}
Q = {surfer, skier}. ____________

30. Make an appropriate choice for the universal set, if the sets under discussion are:
$P = \{x | x$ is a discontented student$\}$
$Q = \{x | x$ is a college freshman$\}$. ____________

31. Make an appropriate choice for the universal set, if the sets under discussion are:
P = {blue jay, cardinal, robin}
Q = {chickadee, pelican}. ____________

32. Let $U = \{x | x$ is a person$\}$ and $A = \{x | x$ is a male$\}$. What is the complement of A? ____________

33. Let $U = \{x | x$ is a person$\}$ and $A = \{x | x$ has blue eyes$\}$. What is the complement of A? ____________

34. Let U = {r, s, t, u, v} and A = {s, t}. Find A'. ____________

35. Let U = {1, 2, 3, 4, 5} and A = {2, 5}. Find A'. ____________

36. Let U = {x, y, z} and A be the set consisting of the last three letters of the English alphabet. Find A'. ____________

37. Let U = {J, F, K} and A be the set consisting of John F. Kennedy's initials. Find A'. ____________

38. Let U = {a, b, c, d}. Find $\emptyset'$. ____________

39. Let $U = \{x | x$ is a mammal$\}$. Find $\emptyset'$. ____________

40. Find the intersection of {a, b, c, d, e} and {c, d, e}. ____________

41. Find the intersection of {u, v, w, x, y, z} and {w, x}. ____________

42. Find $\{1, 2, 3, 4, 5\} \cap \{2, 5\}$. ____________

43. Find $\{7, 8, 9\} \cap \{4, 5, 6, 7\}$. ____________

44. Find $\{1, 2, 3, 4\} \cap \{2, 3\}$. ____________

45. Find $\{7, 8, 9\} \cap \{8\}$. ____________

46. Find the union of {a, b, c, d, e} and {c, d, e}. ____________

47. Find the union of {u, v, w, x, y, z} and {w, x}. ____________

48. Find $\{1, 2, 3, 4, 5\} \cup \{2, 5\}$. ____________

49. Find $\{7, 8, 9\} \cup \{4, 5, 6, 7\}$. ____________

50. Find $\{1, 2, 3, 4\} \cup \{2, 3\}$. ____________

51. Find $\{7, 8, 9\} \cup \{8\}$. ____________

52. Let U = {a, b, c, . . . , j},
A = {d, e, f}, and
B = {g, h, i}.
Find $A \cap B$, $(A \cap B)'$, $A \cup B$, and $(A \cup B)'$.
____

53. Let U = {a, b, c, . . . , l},
A = {d, e, f, g, h}, and
B = {g, h, i, j}.
Find $A \cap B$, $(A \cap B)'$, $A \cup B$, and $(A \cup B)'$.
____

54. Let U = {a, b, c, . . . , z},
A = {a, b, c, . . . , w}, and
B = {a, b, c, . . . , x}.
Find $(A \cup B)'$ and $(A \cap B)'$.
____

55. Let U = {a, b, c, . . . , z},
A = {a, b, c, . . . , v}, and
B = {a, b, c, . . . , w}.
Find $(A \cup B)'$ and $(A \cap B)'$.
____

1-2 INTEGERS; RATIONAL NUMBERS

Many of the sets used in mathematics are sets of numbers. **Numbers** are abstract products of man's imagination. Symbols which represent numbers are called **numerals**. For example, two different symbols frequently used to represent the number twelve are the Roman numeral XII and the Hindu-Arabic numeral 12. The Hindu-Arabic numerals are the numerals most commonly used today. The numbers called zero, one, two, three, four, five, six, seven, eight, and nine are commonly represented by the Hindu-Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The numerals in the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} are called **digits**.

The numbers we use in counting are called the **counting numbers**, the **natural numbers**, or the **positive integers**. They are represented by the Hindu-Arabic numerals 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, etc. Any positive integer can be represented by a numeral composed of one or more of the ten digits written in a horizontal line. For example, seven thousand two hundred fifty-six is represented by 7256. The numeral 7256 represents the number consisting of six ones plus five tens plus two hundreds plus seven thousands.

$$7256 = 6 \text{ ones} + 5 \text{ tens} + 2 \text{ hundreds} + 7 \text{ thousands}$$

Otherwise written,

$$7256 = 7 \text{ thousands} + 2 \text{ hundreds} + 5 \text{ tens} + 6 \text{ ones}.$$

This system of numerals is based on ten and is called the **decimal system**. Reading right to left, the first digit represents a number of ones, and thereafter each digit represents **ten** times the value it would have, if it were located one place to its right. We might say that, reading right to left, the first place-value is one, and thereafter each place-value is ten times the value of the place immediately to its right. This is illustrated in Figure 1.

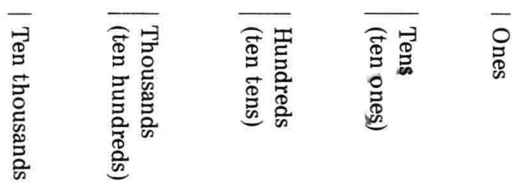

Figure 1 Place-Value in the Decimal System

To make large numbers easier to read, it is customary to insert a comma between the hundreds and thousands place and to insert a comma every third place to the left thereafter. For example,

$$7256 = 7{,}256$$
$$7158023 = 7{,}158{,}023.$$

The set consisting of all the positive integers may be written

$$\{1, 2, 3, 4, \ldots\}.$$

Here the three dots indicate that all the rest of the positive integers are included in the set even though their numerals are not explicitly written. Sometimes, for emphasis, the numeral for a positive integer might be prefixed with a plus sign. That is, the set {1, 2, 3, 4, . . .} might be written

$$\{+1, +2, +3, +4, \ldots\}.$$

The set consisting of the positive integers together with zero, 0, is called **the set of whole numbers** and may be written

$$\{0, 1, 2, 3, 4, \ldots\}.$$

Associated with each positive integer there is a number called its **negative** or its **additive inverse**. The negative of a positive integer is designated by writing the numeral for the positive integer with a negative or minus sign in front of it. For example, the negative of 7 is written -7 and may be read "the negative of seven", "the additive inverse of seven", or simply "minus seven".

The negatives of all the counting numbers constitute **the set of negative integers.** This set may be denoted by

$$\{-1, -2, -3, -4, \ldots\}.$$

The set consisting of all the positive integers, zero, and all the negative integers is called the set of **integers.** The integers may be written

$$\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

The set of integers "evenly divisible" by the integer 2,

$$\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$$

is called the set of **even integers.** The set of integers other than the even integers,

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$

is called the set of **odd integers.**

A **rational number** is any number which can be expressed in the form $\frac{a}{b}$ where a and b are integers and b is not zero. The integer a is called the **numerator**, and the integer b is called the **denominator.**

$$\frac{12}{7}, \frac{-4}{5}, \frac{0}{8}, \frac{3}{1}, \text{ and } \frac{-9}{1} \text{ represent rational numbers.}$$

We can consider the integers to be rational numbers by identifying each integer with a rational number. For example, identify the integer 2 with the rational number $\frac{2}{1}$; identify -5 with $\frac{-5}{1}$; identify 0 with $\frac{0}{1}$.

For any integer a, let $a = \frac{a}{1}$.

The word **fraction** is commonly used for *any* **symbol** of the form $\frac{p}{q}$ where p and q are mathematical expressions such that q is not zero. The p is called the **numerator**, and the q is called the **denominator**. In this sense of the word fraction, any rational number can be written as a fraction.

Two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, are said to be **equal** provided that the products $a \times d$ and $b \times c$ are equal.

Definition $\frac{a}{b} = \frac{c}{d}$ provided that $a \times d = b \times c$.

$$\frac{3}{4} = \frac{6}{8} \text{ because } 3 \times 8 = 4 \times 6.$$

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{c}{d}$ is said to be an **equivalent form** of $\frac{a}{b}$.

In general, two fractions, $\frac{p}{q}$ and $\frac{r}{s}$, are said to be **equal** provided that the products $p \times s$ and $q \times r$ are equal.

The definition of equality of fractions may be used to prove an extremely useful principle.

Fundamental Principle of Fractions If both numerator and denominator of a fraction are multiplied or divided by the same nonzero number, the resulting fraction is equal to the original one.

$$\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20} \qquad \frac{9}{27} = \frac{9 \div 9}{27 \div 9} = \frac{1}{3}$$

A **positive rational number** is any rational number in which the numerator and denominator have the same sign.

$$\frac{2}{3} \text{ and } \frac{-4}{-5} \text{ are positive rational numbers.}$$