CONTINUED FRACTIONS

AND PADÉ APPROXIMANTS

Edited by

C. BREZINSKI

North-Holland

Continued Fractions and Padé Approximants

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C. Brezinski

Université des Sciences et Techniques
de Lille Flandres-Artois
U.F.R. Informatique Electronique
Electrotechnique Automatique
Laboratoire d'Analyse Numérique et d'Optimisation
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INTRODUCTION

Padé approximants and continued fractions are typical examples of old domains (since continued fractions can be traced back at least to Euclid's g.c.d. algorithm more than 2000 years ago) which are now in full vitality. This is due to their numerous applications in number theory, cryptography, statistics, numerical analysis, special functions, digital filtering, signal processing, fractals, fluid mechanics, theoretical physics, chemistry, engineering, etc. This renewal of interest is also due to their intimate connections with other important topics such as orthogonal polynomials (another old subject now again in full vitality), rational approximation, Gaussian quadratures, extrapolation and convergence acceleration methods, solution of differential equations, projection methods for solving systems of linear and nonlinear equations, and so on. A complete bibliography on these domains would contain more than 6000 references, many international conferences took place, and many books were published these past few years.

Since the subject is very rapidly developing, it seemed that a book gathering carefully selected papers presenting the last results would be of interest. This book comes from a special issue of the IMACS journal *Applied Numerical Mathematics* (Vol. 4, Numbers 2–4, June 1988) plus some new contributions specially written for it. All the papers contained in this book are original and important.

I would like to thank all the authors, renowned experts in these fields, for their enthusiastic help.

Claude Brezinksi
Laboratoire d'Analyse Numérique et d'Optimisation
Université des Sciences et Techniques de Lille Flandres-Artois
59655 Villeneuve d'Ascq Cedex, France

CONTENTS

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ON THE ASYMPTOTIC BEHAVIOUR OF CONTINUED FRACTIONS

Claude BREZINSKI
Laboratoire d'Analyse Numérique et d'Optimisation, UFR IEEA - M3, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France

The asymptotic behaviour of ratios of differences of two convergents of a continued fraction is studied. In the first section the case of general continued fractions is considered, while in the second section limit k-periodic ones are treated. Necessary and sufficient conditions are obtained. The behaviour of ratios of the error is also studied and some acceleration methods are given.

Introduction

The asymptotic behaviour of limit periodic continued fractions is well known. In particular Perron proved that they converge linearly [6]. The reciprocal of this result was obtained only recently [1]. In Section 1 of this paper the asymptotic behaviour of general continued fractions will be studied while Section 2 will be devoted to limit k-periodic continued fractions. Surprisingly all the proofs are very elementary, some of them even obvious, and it is, again, a strange situation, often observed in research, that they were not found before. In passing, a straightforward proof of the main result of [1] on limit periodic continued fractions is obtained.

We shall consider continued fractions with complex elements and denominators all equal to one, that is, continued fractions of the form

$$C = b_0 + \frac{a_1}{1} + \frac{a_2}{1-a_2} + \frac{a_3}{1-a_3} + \cdots. \tag{1}$$

This assumption does not restrict the generality since it is always possible to transform a continued fraction into an equivalent one of the previous form if and only if all its partial denominators are different from zero [4].

In the sequel C will always denote the above continued fraction and C_n its convergents. In general, C will not be assumed to converge but, if so, C will be its value.

1. General results

In this section we shall give some general results on the asymptotic behaviour of continued fractions without any special assumption on their elements.

Theorem 1.1. *Let* $0 < p_0 < p_1 < \cdots$ *be an infinite strictly increasing sequence of positive integers, let*

$$C' = b'_0 + \frac{a'_1}{1} + \frac{a'_2}{1-a'_2} + \frac{a'_3}{1-a'_3} + \cdots$$

be the continued fraction with convergents $C'_n = C_{p_n}$, $n = 0, 1, \ldots,$ *and let* $r \in \mathbb{C}$.

A necessary and sufficient condition that

$$\lim_{n\to\infty} \frac{C_{p_{n+1}} - C_{p_n}}{C_{p_n} - C_{p_{n-1}}} = r$$

is that

$$\lim_{n\to\infty} a'_n = -r.$$

Proof. We have $b'_0 = C_{p_0}$, $a'_1 = C_{p_1} - C_{p_0}$, and

$$a'_n = (C_{p_{n-1}} - C_{p_n})/(C_{p_{n-1}} - C_{p_{n-2}}), \quad n = 2, 3, \ldots.$$

Thus the result immediately follows. □

This result does not assume the convergence of the continued fractions C and C'. Using some theorems due to Delahaye [2] we immediately obtain the following results for C'.

Theorem 1.2. *We assume that* $\forall n$, $C'_n \neq C'_{n+1}$.

(i) *Let* $r \in \mathbb{C}$, $|r| \neq 1$. $\exists x \in \mathbb{C}$ *such that* $(C'_{n+1} - x)/(C'_n - x)$ *converges to* r *iff* $\Delta C'_{n+1}/\Delta C'_n$ *converges to* r.

(ii) *Let* $r \in \mathbb{C}$, $|r| < 1$. *If* $\Delta C'_{n+1}/\Delta C'_n$ *converges to* r, *then* (C'_n) *converges to a limit* C' *and* $(C'_{n+1} - C')/(C'_n - C')$ *tends to* r.

(iii) *Let* $r \in \mathbb{C}$, $|r| > 1$. *If* $\Delta C'_{n+1}/\Delta C'_n$ *converges to* r, *then*

$$\lim_{n\to\infty} |C'_n| = +\infty$$

and $\forall x \in \mathbb{C}$, $(C'_{n+1} - x)/(C'_n - x)$ *converges to* r.

Let us remark that the statement (i) does not imply the convergence of (C'_n) to x.

We shall now give a result on the continued fraction C itself. We have the following theorem.

Theorem 1.3. *Let* $0 < p_0 < p_1 < \cdots$ *be an infinite strictly increasing sequence of positive integers and let* $r \in \mathbb{C}$. *A necessary and sufficient condition that*

$$\lim_{n\to\infty} \frac{C_{p_n} - C_{p_n-1}}{C_{p_n-1} - C_{p_n-2}} = r,$$

is that

$$\lim_{n\to\infty} a_{p_n} = -r.$$

Proof. Since the denominators of the convergents of C are all equal to one,

$$C_n = (1 - a_n)C_{n-1} + a_n C_{n-2}, \quad n = 2, 3, \ldots,$$

with $C_0 = b_0$ and $C_1 = b_0 + a_1$.

Therefore,

$$a_n = -\Delta C_{n-1}/\Delta C_{n-2},$$

and the result follows from replacing the index n by p_n. □

2. Limit k-periodic continued fractions

Let us begin with a remark. In Theorem 1.1, if $p_n = n$, then $a'_n = a_n$. Thus a necessary and sufficient condition for $\Delta C_n/\Delta C_{n-1}$ to have a limit is that the continued fraction C be limit periodic. Since the continued fraction is equivalent to

$$b_0 + \frac{a_1/b_1}{1} \, + \, \frac{a_2/b_1 b_2}{1} \, + \, \frac{a_3/b_2 b_3}{1} \, + \, \cdots,$$

with $b_n = 1 - a_n$,

$$\lim_{n\to\infty} a_n/b_{n-1}b_n = -r/(1+r)^2 = a,$$

which is exactly the result given in [1] with a more complicated proof. Moreover, if $a \neq -\frac{1}{4} + c$ with $c \leqslant 0$, the continued fraction converges and $|r| \neq 1$.

Let us now turn to limit k-periodic continued fractions. We recall that if $\exists r_0, \ldots, r_{k-1} \in \mathbb{C}$ such that

$$\lim_{n\to\infty} a_{m+nk} = r_m, \quad m = 0, 1, \ldots, k-1,$$

then the continued fraction C is said to be limit k-periodic and k is its period. Of course $\forall m$, $r_m = r_{m+k}$.

Let us first mention that our assumption on (1) that $\forall n$, $B_n = 1$ does not restrict the generality. Indeed, if we consider the general limit k-periodic continued fraction

$$C = v_0 + \frac{u_1}{v_1} \, + \, \frac{u_2}{v_2} \, + \, \cdots, \tag{2}$$

with

$$\lim_{n\to\infty} u_{m+nk} = u^{(m)},$$

and

$$\lim_{n\to\infty} v_{m+nk} = v^{(m)},$$

for $m = 0, \ldots, k-1$, then, as proved by Lembarki [5],

$$\lim_{n\to\infty} h_{m+nk} = v^{(m)}/(1 + r^{(m)}),$$

where $h_n = B_n/B_{n-1}$ and

$$r^{(m)} = \lim_{n\to\infty} \Delta C_{m+nk-1}/\Delta C_{m+nk-2}.$$

But (2) is equivalent to

$$v_0 + \frac{u_1/v_1}{1} \, + \, \frac{a_2}{1-a_2} \, + \, \frac{a_3}{1-a_3} \, + \, \cdots = v_0 + \frac{d_1 u_1}{d_1 v_1} \, + \, \frac{d_1 d_2 u_2}{d_2 v_2} \, + \, \cdots, \tag{3}$$

with $d_n = 1/h_n$. Thus

$$\lim_{n\to\infty} a_{m+nk} = \lim_{n\to\infty} \frac{u_{m+nk}}{h_{m+nk}h_{m+nk-1}} = u^{(m)}(1+r^{(m)})(1+r^{(m-1)})/v^{(m)}v^{(m-1)},$$

which shows that (3) is limit k-periodic and that $\forall n,\ B_n = 1$. Therefore any general limit k-periodic continued fraction of the form (2) can be transformed into an equivalent limit k-periodic continued fraction of the form (3), and thus all the results stated in this section apply to (2).

The first result on limit k-periodic continued fractions immediately follows by setting $p_n = m + nk$ in Theorem 1.3.

Theorem 2.1. *Let $r_0, \ldots, r_{k-1} \in \mathbb{C}$. A necessary and sufficient condition that*

$$\lim_{n\to\infty} \frac{C_{m+nk} - C_{m+nk-1}}{C_{m+nk-1} - C_{m+nk-2}} = r_m, \quad m = 0, \ldots, k-1,$$

is that

$$\lim_{n\to\infty} a_{m+nk} = -r_m, \quad m = 0, \ldots, k-1.$$

We shall now examine the asymptotic behaviour of some other similar ratios.

Theorem 2.2. *Let C be a limit k-periodic continued fraction. We set*

$$\frac{A_{k-1,m+(n-1)k+1}}{B_{k-1,m+(n-1)k+1}} = 1 - a_{m+(n-1)k+1} + \frac{a_{m+(n-1)k+2}}{1 - a_{m+(n-1)k+2}} + \cdots + \frac{a_{m+nk}}{1 - a_{m+nk}},$$

with $a_0 = 1 - b_0$ and

$$\lim_{n\to\infty} \frac{A_{k-1,m+(n-1)k+1}}{B_{k-1,m+(n-1)k+1}} = \frac{A'_{k-1,m+1}}{B'_{k-1,m+1}}$$

$$= 1 - r_{m+1} + \frac{r_{m+2}}{1 - r_{m+2}} + \cdots + \frac{r_{m+k}}{1 - r_{m+k}}.$$

We assume that $B_{k-1,m+nk+1} \neq 0$ and that $B'_{k-1,m+1}$ exists and is different from zero for $m = 0, \ldots, k-1$. Then, $\forall m$,

$$\lim_{n\to\infty} \frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+nk} - C_{m+(n-1)k}} = (-1)^k r_0 \cdots r_{k-1}.$$

Proof. We know that [6],

$$C_{p_n} - C_{p_{n-1}} = (-1)^{p_{n-1}} a_1 \cdots a_{p_{n-1}+1} B_{p_n - p_{n-1} - 1, p_{n-1}+1}.$$

Thus, for $p_n = m + nk$, we get

$$\frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+nk} - C_{m+(n-1)k}} = (-1)^k a_{m+(n-1)k+2} \cdots a_{m+nk+1} \frac{B_{k-1,m+nk+1}}{B_{k-1,m+(n-1)k+1}}.$$

When n tends to infinity the ratio in the right-hand side tends to one, due to the assumptions, and

$$\lim_{n\to\infty} a_{m+(n-1)k+2} \cdots a_{m+nk+1} = r_{m+2} \cdots r_{m-1} r_m r_{m+1} = r_0 \cdots r_{k-1}. \qquad \square$$

Of course the reciprocal of this theorem cannot be true. For example, if $\exists r \in \mathbb{C}$ such that for $m = 0$ and 1,

$$\lim_{n\to\infty} (C_{m+2n+2} - C_{m+2n})/(C_{m+2n} - C_{m+2n-2}) = r,$$

it can correspond either to a limit 1- or 2-periodic continued fraction. However, different types of complementary results can be obtained. We first have the next theorem.

Theorem 2.3. *Let C be a limit k-periodic continued fraction. Then, $\forall m$,*

$$\lim_{n\to\infty} \frac{C_{m+nk} - C_{m+nk-1}}{C_{m+(n-1)k} - C_{m+(n-1)k-1}} = (-1)^k r_0 \cdots r_{k-1}.$$

Proof. We have

$$\frac{C_{m+nk} - C_{m+nk-1}}{C_{m+(n-1)k} - C_{m+(n-1)k-1}} = \frac{C_{m+nk} - C_{m+nk-1}}{C_{m+nk-1} - C_{m+nk-2}} \cdots \frac{C_{m+(n-1)k+1} - C_{m+(n-1)k}}{C_{m+(n-1)k} - C_{m+(n-1)k-1}},$$

and the result follows from Theorem 2.1. $\square$

A second type of result is the following.

Theorem 2.4. *If $\exists r_0, \ldots, r_{k-1} \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} \frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+(n+1)k-1} - C_{m+nk-1}} = r_m, \quad m = 0, 1, \ldots,$$

then

$$\lim_{n\to\infty} \frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+nk} - C_{m+(n-1)k}} = r_0 \cdots r_{k-1}.$$

Proof. We have

$$\frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+nk} - C_{m+(n-1)k}} = \frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+(n+1)k-1} - C_{m+nk-1}} \cdots \frac{C_{m+nk+1} - C_{m+(n-1)k+1}}{C_{m+nk} - C_{m+(n-1)k}}.$$

When n tends to infinity, this ratio tends to $r_m r_{m-1} \cdots r_{m-k+1} = r_0 \cdots r_{k-1}$. $\square$

If C is a limit k-periodic continued fraction, it is easy to prove that, under the assumptions of Theorem 2.2, $\forall m$,

$$\lim_{n\to\infty} \frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+(n+1)k-1} - C_{m+nk-1}} = -r_m,$$

and thus the result of Theorem 2.2 follows from Theorem 2.4. It must be noticed that the factor $(-1)^k$ is due to the fact that in some theorems r_m is the limit of a_{m+nk} while, in some others, it is the limit of a ratio which, if the continued fraction is limit k-periodic, tends to $-r_m$.

The preceding results do not assume the convergence of the continued fraction C. Using some theorems due to Delahaye [2] we immediately obtain the next one.

Theorem 2.5. *Let* $(r_0,\ldots,r_{k-1}) \in \mathbb{C}^k$ *and* $(q_0,\ldots,q_{k-1}) \in \mathbb{C}^k$ *such that*

(B) $q_0 \neq 1, \quad q_1 \neq 1, \quad \ldots, \quad q_{k-1} \neq 1,$

(L) $|r_0 r_1 \cdots r_{k-1}| \neq 1,$

$$1 + r_0 + r_0 r_1 + \cdots + r_0 r_1 \cdots r_{k-2} \neq 0,$$
$$1 + r_1 + r_1 r_2 + \cdots + r_1 r_2 \cdots r_{k-1} \neq 0,$$
$$\vdots$$
$$1 + r_{k-1} + r_{k-1} r_0 + \cdots + r_{k-1} r_0 \cdots r_{k-3} \neq 0$$

with

(LB) $r_0 = q_0 \dfrac{q_1 - 1}{q_0 - 1},$

$$r_1 = q_1 \frac{q_2 - 1}{q_1 - 1}, \quad \ldots, \quad r_{k-2} = q_{k-2} \frac{q_{k-1} - 1}{q_{k-2} - 1}, \qquad r_{k-1} = q_{k-1} \frac{q_0 - 1}{q_{k-1} - 1},$$

(BL) $q_0 = \dfrac{r_0 + r_0 r_1 + \cdots + r_0 r_1 \cdots r_{k-1}}{1 + r_0 + r_0 r_1 + \cdots + r_0 r_1 \cdots r_{k-2}},$

$$q_1 = \frac{r_1 + r_1 r_2 + \cdots + r_1 r_2 \cdots r_0}{1 + r_1 + r_1 r_2 + \cdots + r_1 r_2 \cdots r_{k-1}},$$
$$\vdots$$
$$q_{k-1} = \frac{r_{k-1} + r_{k-1} r_0 + \cdots + r_{k-1} r_0 \cdots r_{k-2}}{1 + r_{k-1} + r_{k-1} r_0 + \cdots + r_{k-1} r_0 \cdots r_{k-3}}.$$

(i) $\exists x \in \mathbb{C}$ *such that for* $m = 0, 1, \ldots, k-1,$

$$\lim_{n\to\infty} (C_{m+nk+1} - x)/(C_{m+nk} - x) = q_m,$$

iff, for $m = 0, \ldots, k-1,$

$$\lim_{n\to\infty} (C_{m+nk+2} - C_{m+nk+1})/(C_{m+nk+1} - C_{m+nk}) = r_m.$$

(ii) *If* $\exists r_0, \ldots, r_{k-1} \in \mathbb{C}$ *satisfying* (L) *and such that* $|r_0 r_1 \cdots r_{k-1}| < 1$, *and if, for* $m = 0, 1, \ldots, k-1,$

$$\lim_{n\to\infty} (C_{m+nk+2} - C_{m+nk+1})/(C_{m+nk+1} - C_{m+nk}) = r_m,$$

then (C_n) *converges to a limit* C *and, for* $m = 0, \ldots, k-1,$

$$\lim_{n\to\infty} (C_{m+nk+1} - C)/(C_{m+nk} - C) = q_m,$$

with the q_m *given by* (BL).

(iii) *If* $\exists r_0, \ldots, r_{k-1} \in \mathbb{C}$ *satisfying* (L) *and such that* $|r_0 r_1 \cdots r_{k-1}| > 1$, *and if, for* $m = 0, 1, \ldots, k-1,$

$$\lim_{n\to\infty} (C_{m+nk+2} - C_{m+nk+1})/(C_{m+nk+1} - C_{m+nk}) = r_m,$$

then

$$\lim_{n\to\infty} |C_n| = +\infty,$$

and $\forall x \in \mathbb{C}$, $\forall m = 0, 1, \ldots, k-1$,

$$\lim_{n\to\infty} (C_{m+nk+1} - x)/(C_{m+nk} - x) = q_m,$$

with the q_m *given by* (BL).

Due to the preceding results we see that limit k-periodic continued fractions can be accelerated by applying Aitken's Δ^2-process to the subsequences $(C_{m+nk})_n$ for $m = 0, 1, \ldots, k-1$ or, equivalently, by the transformation

$$C_n^* = C_{n+2k} - \frac{(C_{n+2k} - C_{n+k})^2}{C_{n+2k} - 2C_{n+k} + C_n}, \quad n = 0, 1, \ldots.$$

The continued fraction (1) can also be accelerated by a so-called modification of the form

$$S_n(w_n) = \frac{A_n + w_n A_{n-1}}{B_n + w_n B_{n-1}} = \frac{h_n C_n + w_n C_{n-1}}{h_n + w_n}, \quad n = 0, 1, \ldots.$$

In our particular case we have $\forall n$, $h_n = 1$, and

$$\frac{S_n(w_n) - C}{C_{n-1} - C} = \frac{(C_n - C)/(C_{n-1} - C) + w_n}{1 + w_n}.$$

Thus $(S_n(w_n))$ will converge faster than (C_{n-1}) if the numerator of this ratio tends to zero but not the denominator.

If

$$\lim_{n\to\infty} a_{m+nk} = a^{(m)}$$

for $m = 0, \ldots, k-1$, then, by Theorem 2.1,

$$\lim_{n\to\infty} \Delta C_{m+nk-1}/\Delta C_{m+nk-2} = -a^{(m)},$$

and, by Theorem 2.5,

$$\lim_{n\to\infty} (C_{m+nk} - C)/(C_{m+nk-1} - C) = q_{m-1},$$

with

$$q_{m-1} = \frac{-a^{(m+1)} + a^{(m+1)}a^{(m+2)} - \cdots + (-1)^k a^{(m+1)} \cdots a^{(m+k)}}{1 - a^{(m+1)} + a^{(m+1)}a^{(m+2)} - \cdots + (-1)^{k+1} a^{(m+1)} \cdots a^{(m+k-1)}}.$$

Thus, if we take

$$w_{m+nk} =$$

$$= -\frac{\dfrac{\Delta C_{m+nk}}{\Delta C_{m+nk-1}} + \dfrac{\Delta C_{m+nk}}{\Delta C_{m+nk-1}} \dfrac{\Delta C_{m+nk+1}}{\Delta C_{m+nk}} + \cdots + \dfrac{\Delta C_{m+nk}}{\Delta C_{m+nk-1}} \cdots \dfrac{\Delta C_{m+nk+k-1}}{\Delta C_{m+nk+k-2}}}{1 + \dfrac{\Delta C_{m+nk}}{\Delta C_{m+nk-1}} + \dfrac{\Delta C_{m+nk}}{\Delta_{m+nk-1}} \dfrac{\Delta C_{m+nk+1}}{\Delta C_{m+nk}} + \cdots + \dfrac{\Delta C_{m+nk}}{\Delta C_{m+nk-1}} \cdots \dfrac{\Delta C_{m+nk+k-2}}{\Delta C_{m+nk+k-3}}}$$

$$= -\frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+(n+1)k-1} - C_{m+nk-1}},$$

then

$$\lim_{n\to\infty} w_{m+nk} = -\lim_{n\to\infty} (C_{m+nk} - C)/(C_{m+nk-1} - C),$$

and the convergence is accelerated since, by the conditions (B) of Theorem 2.5, this limit is different from -1.

Finally, for limit k-periodic continued fractions, we have obtained the following theorem.

Theorem 2.6. *Let* $C_n^* = C_{n+k+1} - \Delta C_{n+k}(C_{n+k+1} - C_{n+1})/(\Delta C_{n+k} - \Delta C_n)$, $n = 0, 1, \ldots$ *Under the assumptions* (ii) *of Theorem* 2.5, (C_n^*) *converges faster than* (C_n).

For $k = 1$, this transformation reduces to Aitken's Δ^2-process. Otherwise, it is the so-called T_{+k} transformation [3].

Remark 2.7. We have [6]

$$C_{m+(n+1)k} = C_{m+nk} A_{k-1,m+nk+1} + a_{m+nk+1} C_{m+nk-1} B_{k-1,m+nk+1},$$

and

$$1 = A_{k-1,m+nk+1} + a_{m+nk+1} B_{k-1,m+nk+1}.$$

Thus,

$$-a_{m+nk+1} B_{k-1,m+nk+1} = \frac{C_{m+(n+1)k} - C_{m+nk}}{C_{m+nk} - C_{m+nk-1}}.$$

The ratio in the right-hand side is equal to

$$\frac{C_{m+(n+1)k} - C_{m+(n+1)k-1}}{C_{m+(n+1)k-1} - C_{m+(n+1)k-2}} \frac{C_{m+(n+1)k-1} - C_{m+(n+1)k-2}}{C_{m+(n+1)k-2} - C_{m+(n+1)k-3}} \cdots \frac{C_{m+nk+1} - C_{m+nk}}{C_{m+nk} - C_{m+nk-1}}$$
$$+ \frac{C_{m+(n+1)k-1} - C_{m+(n+1)k-2}}{C_{m+(n+1)k-2} - C_{m+(n+1)k-3}} \frac{C_{m+(n+1)k-2} - C_{m+(n+1)k-3}}{C_{m+(n+1)k-3} - C_{m+(n+1)k-4}} \cdots \frac{C_{m+nk+1} - C_{m+nk}}{C_{m+nk} - C_{m+nk-1}}$$
$$+ \cdots + \frac{C_{m+nk+1} - C_{m+nk}}{C_{m+nk} - C_{m+nk-1}}.$$

Thus if $\forall m$,

$$\lim_{n\to\infty} a_{m+nk} = -r_m,$$

then, by Theorem 2.1, this expression tends to

$$\underbrace{r_m r_{m-1} \cdots r_{m+1}}_{k \text{ factors}} + \underbrace{r_{m-1} r_{m-2} \cdots r_{m+1}}_{k-1 \text{ factors}} + \cdots + \underbrace{r_{m+1}}_{1 \text{ factor}}.$$

Moreover,

$$\lim_{n\to\infty} -a_{m+nk+1} B_{k-1,m+nk+1} = r_{m+1} B'_{k-1,m+1}.$$

Thus, if $r_{m+1} \neq 0$, $\forall m$ we have

$$B'_{k-1,m+1} = 1 + r_{m+2} + r_{m+2} r_{m+3} + \cdots + r_{m+2} \cdots r_{m-1} r_m,$$

which shows that, due to the periodicity of the r_m, the conditions $\forall m$, $B'_{k-1,m+1} \neq 0$ of Theorem 2.2 are equivalent to the last k conditions (L) of Theorem 2.5.

Thus, the result of Theorem 2.2 holds under the assumptions that $r_0 \cdots r_{k-1} \neq 0$, and that the last k conditions (L) are satisfied. The condition $B_{k-1,m+nk+1} \neq 0$ insures the existence of the ratios under consideration and it can be removed since, otherwise, the result stated would have no meaning.

To end let us mention that since, for (1), we have

$$C_n = b_0 + a_1 - a_1 a_2 + \cdots + (-1)^{n-1} a_1 \cdots a_n,$$

and

$$\Delta C_n = -a_{n+1} \Delta C_{n-1},$$

the usual convergence test for series can be used to obtain convergence conditions for the continued fraction. In particular the d'Alembert's test says that if $\exists K < 1$, $\exists N$, $\forall n \geqslant N$, $|a_n| \leqslant K$, then (C_n) converges.

References

[1] C. Brezinski and A. Lembarki, The linear convergence of limit periodic continued fractions, *J. Comput. Appl. Math. 19* (1987) 75–77.

[2] J.P. Delahaye, *Theory of Sequence Transformations in Numerical Analysis* (Springer, Heidelberg, 1988).

[3] H.L. Gray and W.D. Clark, On a class of nonlinear transformations and their applications to the evaluation of infinite series, *J. Res. Nat. Bur. Standards B 73* (1969) 251–274.

[4] A. Lembarki, Acceleration of limit periodic continued fractions by the T_{+M} transformation, *J. Comput. Appl. Math. 19* (1987) 109–116.

[5] A. Lembarki, Convergence acceleration of limit k-periodic continued fractions, *Appl. Numer. Math. 4* (1988) 337–349.

[6] O. Perron, *Die Lehre von den Kettenbrüchen* (Teubner, Stuttgart, 1913).

Continued Fractions and Padé Approximants, 11–51
C. Brezinski, ed.

A MATRIX EUCLIDEAN ALGORITHM AND THE MATRIX MINIMAL PADÉ APPROXIMATION PROBLEM

Adhemar BULTHEEL and Marc VAN BAREL
K.U. Leuven, Department of Computer Science, Celestijnenlaan 200A, B-3030 Leuven (Heverlee), Belgium

We formulate a minimal Padé approximation problem and generalize it for rectangular matrix series. The scalar solutions can be computed by the Euclidean algorithm and we give its generalization to compute the matrix approximants. The algorithm is normalized so that it generates unique matrix minimal Padé approximants and a unique continued fraction representation for them.

Keywords: Matrix Padé approximation, rational matrices, matrix Euclidean algorithm, block Hankel matrices.

Introduction

The problem of Padé approximation is well known for scalar series $F(z)$. If we allow the series to be $p \times m$ matrix series, it is not trivial to extend the theory. To give an idea of the problems that may arise, we note the following.

A scalar rational form $N(z)M(z)^{-1}$ can be simply normalized by requiring e.g. that $M(z)$ is monic. If however $N(z) \in \mathbb{C}^{p\times m}[z]$ and $M(z) \in \mathbb{C}^{m\times m}[z]$, the normalization is not simple. We have chosen for the simplest possible (in a certain sense) representation of $M(z)$, i.e. for a canonical form of $M(z)$.

Another problem which arises in the matrix case is the degree of the numerator and denominator which is needed in the definition of Padé approximant. In the scalar case, there is only one degree for the numerator and one for the denominator. For the matrix case, one could consider all possible kinds of degree information since each entry of the numerator and the denominator matrix is a scalar polynomial with a specific degree. This seems to be too detailed a specification of the degrees in the general matrix case, although it works for vector approximants. The other extreme is to consider the numerator and denominator as polynomials with matrix coefficients. In that case we associated again one degree with the numerator and one degree with the denominator. This seems to work partially for the square matrix case when we have a normal Padé table. However, a matrix coefficient can be singular without being zero and these situations of the nonnormal case ask for more detailed degree information. We shall here consider row degrees for the matrices. This is something in between the two previous extremes. Thus we could write $N(z) \in (\mathbb{C}^{m}[z])^{p}$ rather than $N(z) \in \mathbb{C}^{p\times m}[z]$, but we shall not do so. It is then a logical consequence to consider the orders of approximation also rowwise.

Our approach will be via the Euclidean algorithm. It is known that, in the scalar case, it can be used to generate Padé approximants found along an antidiagonal of the Padé table; i.e. with at least a certain fixed order of approximation and with increasing denominator degree and