An Introduction to Lagrangian Mechanics

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To the memory of my father

Yvon Brizard (1929-2007)

Preface

The structure of the present lecture notes on the Lagrangian mechanics of particles and fields is based on achieving several goals. As a first goal, I wanted to model these notes after the wonderful monograph of Landau and Lifschitz on *Mechanics* [12], which is often thought to be too concise for most undergraduate students. One of the many positive characteristics of Landau and Lifschitz's *Mechanics*, however, is that Lagrangian mechanics is introduced in its first chapter and not in later chapters as is usually done in more standard textbooks used at the sophomore/junior undergraduate level.[1] Consequently, the Lagrangian method becomes the centerpiece of the present course and provides a continuous thread throughout the text. This course has been taught at Dartmouth College and Saint Michael's College in approximately the same format proposed in these lecture notes.

As a second goal, the lecture notes introduce several numerical investigations of dynamical equations appearing throughout the text. These numerical investigations present an interactive pedagogical approach, which should enable students to begin their own numerical investigations. As a third goal, an attempt was made to introduce historical facts (whenever appropriate) about the pioneers of Classical Mechanics. Much of the historical information included in the Notes is taken from excellent books by René Dugas [4], Wolfgang Yourgrau and Stanley Mandelstam [18], and Cornelius Lanczos [11]. In fact, from a pedagogical point of view, this historical perspective helps educating undergraduate students in establishing the deep connections between Classical and Quantum Mechanics, which are often ignored or even inverted (as can be observed when students are surprised

[1]The reader is invited to read *A call to action* by E. F. Taylor [Am. J. Phys. **71**, 423-425 (2003)], which promotes a reorganization of undergraduate physics education that includes an early introduction of Lagrangian Mechanics (the Principle of Least Action) into the physics curriculum.

to learn that Hamiltonians have an independent classical existence). As a fourth and final goal, I wanted to keep the scope of these notes limited to a one-semester course in contrast to standard textbooks, which often include an extensive review of Newtonian Mechanics as well as additional material such as Hamiltonian chaos.

It is expected that students taking this course will have had a one-year calculus-based introductory physics course followed by a one-semester course in Modern Physics. Ideally, students should have completed their full calculus sequence and, perhaps, have taken a course on ordinary differential equations. On the other hand, this course should be taken before a rigorous course in Quantum Mechanics in order to provide students with a sound historical perspective involving the connection between Classical Physics and Quantum Physics. Hence, the fall semester of the junior year provides a perfect niche for this course. Topics identified with an asterisk can also be included in a more advanced course.

The standard topics covered in these notes are: The Calculus of Variations (Chapter 1), Lagrangian Mechanics (Chapter 2), Hamiltonian Mechanics (Chapter 3), Motion in a Central Field (Chapter 4), Collisions and Scattering Theory (Chapter 5), Motion in a Non-Inertial Frame (Chapter 6), Rigid Body Motion (Chapter 7), Normal-Mode Analysis (Chapter 8), and Continuous Lagrangian Systems (Chapter 9). Each chapter contains a set of problems with variable level of difficulty. Lastly, in order to ensure a self-contained presentation, a summary of mathematical methods associated with linear algebra and numerical analysis is presented in Appendix A. Appendix B presents a brief introduction to the applications of the Jacobi and Weierstrass elliptic functions in Classical Mechanics; see Whittaker's textbook [17] for many more applications. Lastly, Appendix C presents a brief summary of differential geometric methods in the modern formulation of Hamiltonian mechanics and perturbation theory.

Several innovative topics not normally discussed in standard undergraduate textbooks are included throughout the notes. In Chapter 1, a complete discussion of Fermat's Principle of Least Time is presented, from which a generalization of Snell's Law for light refraction through a nonuniform medium is derived and the equations of geometric optics are obtained [3]. We note that Fermat's Principle proves to be an ideal introduction to variational methods in the undergraduate physics curriculum since students are already familiar with Snell's Law of light refraction.

In Chapter 2, we establish the connection between Fermat's Principle of Least Time and Maupertuis-Jacobi's Principle of Least Action. In particular, Jacobi's Principle introduces a geometric representation of single-particle dynamics that establishes a clear pre-relativistic connection between Geometry and Physics. Next, the nature of mechanical forces (e.g., active versus passive forces) is discussed within the context of d'Alembert's Principle, which is based on a dynamical generalization of the Principle of Virtual Work. Lastly, the fundamental link between the energy-momentum conservation laws and the symmetries of the Lagrangian function is first discussed through Noether's Theorem and then Routh's procedure to eliminate ignorable coordinates is applied to a Lagrangian with symmetries.

In Chapter 3, we present a brief discussion of Hamiltonian optics and the wave-particle duality that established the connection between Classical Physics and Quantum Physics. The problem of charged-particle motion in an electromagnetic field is also investigated by the Lagrangian method in the three-dimensional configuration space and the Hamiltonian method in six-dimensional phase space. This important physical example presents a clear link between the Lagrangian and Hamiltonian methods. In Chapter 4, we discuss the role of the Laplace-Runge-Lenz vector invariant in determining the shape of the Kepler bounded orbit. We also use the Laplace-Runge-Lenz vector to study the precession of a perturbed Keplerian orbit. In Chapter 5, we present a complete solution of the soft-sphere scattering problem as well as the problem of elastic scattering by a hard surface. In Chapter 9, we present the variational derivations of the Schroedinger equation and the Euler equations for a perfect fluid. Using the Noether method, we also derive their respective conservation laws.

In Appendix B, we present an introduction to the applications of the Jacobi and Weierstrass elliptic functions in Classical Mechanics. These interesting functions used to be part of the standard curriculum in Classical Mechanics [17, 12] and have now all but disappeared from modern textbooks [7, 13]. For the Jacobi elliptic function, we consider the problems of motion in a quartic potential, while for the Weierstrass elliptic function, we consider the problem of motion in a cubic potential. The problem of the planar pendulum is used to establish the connection between the Jacobi and Weierstrass elliptic functions. Lastly, in Appendix C, we present a brief introduction to noncanonical Hamiltonian mechanics and canonical Hamiltonian perturbation theory.

My interest in Lagrangian Mechanics was awakened more than 30 years ago when I was an undergraduate student at the Collège Militaire Royal de Saint Jean (Canada). One of my professors (Fernand Ledoyen) bravely taught me Lagrangian Mechanics with Landau and Lifschitz [12] and Arnold

[1] as our constant companions. I remember being immediately struck by the beauty of Lagrangian Mechanics and the power of its methods. I have used Lagrangian methods in my own research in plasma physics for the past 20 years. I would like to thank my *Lagrangian* collaborators Allan N. Kaufman (University of California at Berkeley) and Eugene (Gene) R. Tracy (College of William and Mary) for their friendship and support during this time.

Lastly, I owe a great debt of love and gratitude to my wife (Dinah Larsen) and son (Peter Brizard Larsen) and I thank them for their patience and understanding during the arduous process of writing this book.

Alain Jean Brizard

Contents

Chapter 1

The Calculus of Variations

A wide range of equations in physics, from quantum field and superstring theories to general relativity, from fluid dynamics to plasma physics and condensed-matter theory, are derived from action (variational) principles [2, 15]. The purpose of this Chapter is to introduce the methods of the Calculus of Variations that figure prominently in the formulation of action principles in physics.

1.1 Foundations of the Calculus of Variations

1.1.1 *A Simple Minimization Problem*

It is a well-known fact that the shortest distance between two points in Euclidean space is calculated along a straight line joining the two points. Although this fact is intuitively obvious, we begin our discussion of the problem of minimizing certain integrals in mathematics and physics with a search for an explicit proof. In particular, we prove that the straight line $y_0(x) = mx$ yields a path of shortest distance between the two points $(0,0)$ and $(1,m)$ on the (x,y)-plane as follows.

First, we consider the length integral

$$\mathcal{L}[y] \;=\; \int_0^1 \sqrt{1 + (y')^2}\, dx, \tag{1.1}$$

where $y' = y'(x)$ and the notation $\mathcal{L}[y]$ is used to denote the fact that the value of the integral (1.1) depends on the choice we make for the function $y(x)$; thus, $\mathcal{L}[y]$ is called a *functional* of y. We insist, however, that the function $y(x)$ satisfy the boundary conditions $y(0) = 0$ and $y(1) = m$. Next, we introduce the modified function

$$y(x;\epsilon) \;=\; y_0(x) \;+\; \epsilon\, \delta y(x),$$

where $y_0(x) = mx$ and the variation function $\delta y(x)$ is required to satisfy the prescribed boundary conditions $\delta y(0) = 0 = \delta y(1)$. We thus define the modified length integral

$$\mathcal{L}[y_0 + \epsilon\,\delta y] \;=\; \int_0^1 \sqrt{1 + (m + \epsilon\,\delta y')^2}\,dx$$

as a function of ϵ and a functional of δy. We now show that the function $y_0(x) = mx$ minimizes the integral (1.1) by evaluating the following derivatives

$$\begin{aligned}\left(\frac{d}{d\epsilon}\mathcal{L}[y_0 + \epsilon\,\delta y]\right)_{\epsilon=0} &= \frac{m}{\sqrt{1+m^2}}\int_0^1 \delta y'\,dx \\ &= \frac{m}{\sqrt{1+m^2}}\,[\,\delta y(1) - \delta y(0)] \;=\; 0,\end{aligned}$$

and

$$\left(\frac{d^2}{d\epsilon^2}\mathcal{L}[y_0 + \epsilon\,\delta y]\right)_{\epsilon=0} \;=\; \int_0^1 \frac{(\delta y')^2}{(1+m^2)^{3/2}}\,dx \;>\; 0,$$

which holds for a fixed value of m and all variations $\delta y(x)$ that satisfy the conditions $\delta y(0) = 0 = \delta y(1)$. Hence, we have shown that $y(x) = mx$ minimizes the length integral (1.1) since the first derivative (with respect to ϵ) vanishes at $\epsilon = 0$, while its second derivative is positive at $\epsilon = 0$. We note, however, that our task was made easier by our knowledge of the actual minimizing function $y_0(x) = mx$; without this knowledge, we would be required to choose a trial function $y_0(x)$ and test for all variations $\delta y(x)$ that vanish at the integration boundaries.

Another way to tackle this minimization problem is to find a way to characterize the function $y_0(x)$ that minimizes the length integral (1.1), for *all* variations $\delta y(x)$, without actually solving for $y(x)$. For example, the characteristic property of a straight line $y(x)$ is that its second derivative vanishes for all values of x. The methods of the Calculus of Variations introduced in this Chapter present a mathematical procedure for transforming the problem of minimizing an integral to the problem of finding the solution to an ordinary differential equation for $y(x)$. The mathematical foundations of the Calculus of Variations were developed by Leonhard Euler (1707-1783) and Joseph-Louis Lagrange (1736-1813), who developed the mathematical method for finding curves that minimize (or maximize) certain integrals.

1.1.2 Methods of the Calculus of Variations

1.1.2.1 *Euler's First Equation*

The methods of the Calculus of Variations transform the problem of minimizing (or maximizing) an integral of the form

$$\mathcal{F}[y] \;=\; \int_a^b F(y, y'; x)\, dx \tag{1.2}$$

(with fixed boundary points a and b) into the solution of a differential equation for $y(x)$ expressed in terms of derivatives of the integrand $F(y, y'; x)$, which is assumed to be a smooth function of $y(x)$ and its first derivative $y'(x)$, with a possible explicit dependence on x.

The problem of *extremizing* the integral (1.2) will be treated in analogy with the problem of finding the extremal value of any (smooth) function $f(x)$, i.e., finding the value x_0 such that

$$f'(x_0) \;=\; \lim_{\epsilon\rightarrow 0}\, \frac{1}{\epsilon}\left(f(x_0+\epsilon) \;-\; f(x_0)\right) \;\equiv\; \frac{1}{h}\left(\frac{d}{d\epsilon}f(x_0+\epsilon\, h)\right)_{\epsilon=0} \;=\; 0,$$

where $h \neq 0$ is an arbitrary constant factor.[1] First, we introduce the first-order *functional variation* $\delta\mathcal{F}[y;\delta y]$ defined as

$$\begin{aligned}\delta\mathcal{F}[y;\delta y] &\equiv \left(\frac{d}{d\epsilon}\mathcal{F}[y+\epsilon\,\delta y]\right)_{\epsilon=0} \\ &= \left[\,\frac{d}{d\epsilon}\left(\int_a^b F\left(y+\epsilon\,\delta y, y'+\epsilon\,\delta y', x\right)\, dx\right)\right]_{\epsilon=0},\end{aligned} \tag{1.3}$$

where $\delta y(x)$ is an arbitrary smooth variation of the path $y(x)$ subject to the boundary conditions $\delta y(a) = 0 = \delta y(b)$, i.e., the end points of the path are not affected by the variation (see Fig. 1.1). By performing the ϵ-derivatives in the functional variation (1.3), which involves partial derivatives of $F(y, y', x)$ with respect to y and y', we find

$$\delta\mathcal{F}[y;\delta y] \;=\; \int_a^b \left[\,\delta y(x)\,\frac{\partial F}{\partial y(x)} \;+\; \delta y'(x)\,\frac{\partial F}{\partial y'(x)}\,\right] dx,$$

[1]An *extremum* point refers to either the minimum or maximum of a one-variable function. A *critical* point, on the other hand, refers to a point where the gradient of a multi-variable function vanishes. Critical points include minima and maxima as well as saddle points (where the function exhibits maxima in some directions and minima in other directions). A function $y(x)$ is said to be a *stationary* solution of the functional (1.2) if the first variation (1.3) vanishes for all variations δy that satisfy the boundary conditions.