Johan Grasman

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Asymptotic Methods for Relaxation Oscillations and Applications

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Dedicated to Ank, Laurens and Stefan

Preface

In various fields of science, notably in physics and biology, one is confronted with periodic phenomena having a remarkable temporal structure: it is as if certain systems are periodically reset in an initial state. A paper of Van der Pol in the Philosophical Magazine of 1926 started up the investigation of this highly nonlinear type of oscillation for which Van der Pol coined the name “relaxation oscillation”.

The study of relaxation oscillations requires a mathematical analysis which differs strongly from the well-known theory of almost linear oscillations. In this monograph the method of matched asymptotic expansions is employed to approximate the periodic orbit of a relaxation oscillator. As an introduction, in chapter 2 the asymptotic analysis of Van der Pol’s equation is carried out in all detail. The problem exhibits all features characteristic for a relaxation oscillation. From this case study one may learn how to handle other or more generally formulated relaxation oscillations. In the survey special attention is given to biological and chemical relaxation oscillators.

In chapter 2 a general definition of a relaxation oscillation is formulated. Essential is the existence of a phase of rapid change which can be related to the presence of a small parameter in the equation. An investigation of chaotic and stochastic oscillations completes the analysis of free oscillations of chapter 2. In chapter 3 the dynamics of coupled oscillators is analyzed. The coupling leads to entrainment phenomena for which one may find many applications in biology. The existence of phase waves in a system of spatially distributed coupled oscillators is an example of such an application. In chapter 4 subharmonic and chaotic solutions of the Van der Pol equation with a sinusoidal forcing term are constructed. This problem, formulated by Littlewood, intrigued mathematicians over the last forty years. The horse-shoe mapping of Smale was intended to be used in the analysis of this problem. Later on it was noticed that chaos is present at a much wider scale.

My approach to the analysis of relaxation oscillations is one of an applied mathematician. New developments in the theory of nonlinear dynamical systems made it possible to describe qualitatively phenomena such as chaotic dynamics of physical and biological systems. The question arose whether in such cases quantitative approximations can be made. Using powerful tools from the theory of singular perturbations we came into the position to construct matched asymptotic approximations and to relate them to the results of the qualitative theories. For biologists and physists it is worthy to get acquainted with the outcome of these investigations: entrainment phenomena in systems of coupled biological oscillators can be quantified and chaotic dynamics of driven oscillators can be computed!

I have attempted to give a survey of the literature following the historical developments in the field and to make the bibliography as complete as possible. The book gives an overview of the work that has been done in this field over the last sixty years. There may be ommisions and the contents is perhaps somewhat out of in balance when I deal with the contributions of our group in the Netherlands. One last remark has to be made about the historical element. While reading the papers of Van der Pol, I got impressed by his intuitive judgement of the importance of certain physical phenomena, his ability to analyse them mathematically and by his charming and effective ways of conveying his observations to an audience of scientists and medical investigators.

This study of relaxation oscillations will aquaint the reader with the modeling of periodic phenomena in physics and biology. It, moreover, demonstrates how the modern theory of dynamical systems is applied to a particular type of nonlinear oscillation. In order to explore the distinction between chaos and noise the effect of stochastic perturbations upon the oscillator and its period will be analyzed. Obviously, this broad approach with a diversity of mathematical techniques makes it difficult to study in depth all theories that are brought up. The reader should consult the literature if he wishes to acquire a better background knowledge of a mathematical topic such as the theory of stochastic differential equations or the use of symbolic dynamics.

This book can be read by one who has some basic knowledge of differential equations and asymptotic methods. In the appendices I reviewed concepts of these theories, which are useful to have at hand. The presentation of this material is too brief to use it as an introduction in this field. As a compensation appropriate references are given.

Biologists may, in particular, be interested in chapters 1,2 and 3 with exclusion of sections 2.1.3-2.1.5, 2.2.2-2.2.4 and 3.2.1-3.2.3. Indeed, the book can be read in this way.

I am indebted to Wiktor Eckhaus Michael Ghil, Hans Lauwerier, Jos Roerdinck, Nico Temme and Ferdinand Verhulst for their useful suggestions which have helped to improve the content and clarity of the manuscript. Moreover, I am grateful to the Centre of Mathematics and Computer Science, that gave me the opportunity to carry out the mathematical investigations which has led to the present monograph. The Centre also provided me with all facilities for computing and for reproduction of text and figures. I wish to mention the assistance of Boudewijn de Kerf in the use of computer graphics. Miss Mini Middelberg and miss Sandra Dorrestijn took care of the typing of the manuscript; they did an excellent job. The figures with perspective has been

drawn by Tobias Baanders; his fine style will certainly help the reader to perceive the course of trajectories in phase space.

Amsterdam, August 1986 Johan Grasman

Contents

Appendices

1. INTRODUCTION

Intuitively the dynamics of a relaxation oscillation is easily understood from a simple mechanical system as the see-saw of fig. 1.0.1 with a water reservoir at one side. As the amount of water exceeds the weight at the other side, the see-saw flips. Then the reservoir is emptied and returns to its original position. In the applied sciences relaxation oscillations are most frequently met in biochemical and biological systems. A similar phenomenon is observed for these systems: during a short time interval of the cycle one or more components of the biological system may exhibit a fast change in their density.

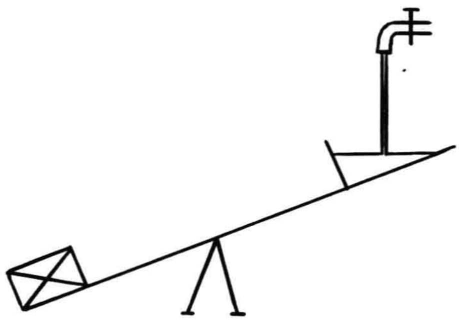


Fig.1.0.1 A typical relaxation oscillator: the see-saw with a water reservoir at one side. The period depends on the rate of inflow and the volume of water in the state of balance.

For almost linear oscillators a more or less generally accepted mathematical theory exists, see Bogoliubov and Mitropolsky (1961). A one degree of freedom system takes in that case the form

$$\frac{d^2x}{dt^2}+x = \epsilon f(x,\frac{dx}{dt}), \quad 0<\epsilon<<1. \tag{1.0.1}$$

When ϵ is not small this theory cannot be applied. In general, no asymptotic approximations can be constructed and solutions have to be found numerically. However, there is an exception: for systems of the type

$$\epsilon\frac{d^2x}{dt^2} = f(x,\frac{dx}{dt}), \quad 0<\epsilon<<1 \tag{1.0.2}$$

again asymptotic approximations can be made. Letting $\epsilon\to 0$ we obtain the equation

$$f(x,\frac{dx}{dt}) = 0, \tag{1.0.3}$$

which by itself cannot have a periodic solution. However, it may be a good approximation of the periodic solution of (1.0.2) during a large time interval of the cycle. Then as a threshold is reached and this approximation breaks down the variable x changes rapidly for a short time interval. From the see-saw of fig. 1.0.1 it is clear which states the system rapidly passes before it returns in a state for which the reduced approximation (1.0.3) holds again. As we will see such a conclusion can also be made from the vector field in state space for other types of relaxation oscillators.

Instead of using qualitative arguments about the phase of rapid change, one may as well construct an asymptotic solution for this time interval by stretching the time variable. This solution must match the one for the phase in which approximation (1.0.3) holds. This technique is known as the method of matched asymptotic expansions. It originates from fluid mechanics, where it is used to analyse boundary layer phenomena. In the last ten years a general theory for this class of problems evolved, see Kevorkian and Cole (1981), O'Malley (1982) and Eckhaus (1979). Characteristic for problems in singular perturbation theory is the small parameter that multiplies the highest derivative, see formula (1.0.2).

In the following sections of this introduction we give an overview of the occurrence of relaxation oscillations in the applied science. We distinguish between free, forced and coupled oscillations, which comes up as a quite natural classification. In the following chapters these oscillations are analyzed mathematically and reference is made to the applications mentioned in this introduction. Moreover, some examples are worked out.

1.1 The Van der Pol oscillator

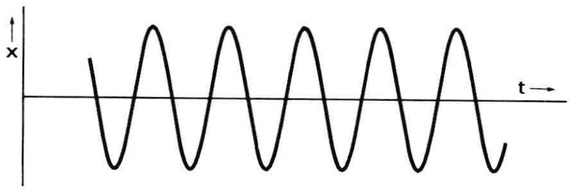
Relaxation oscillations were observed the first time by Van der Pol (1926), who studied properties of a triode circuit. Such a system exhibits self-sustained oscillations with an amplitude independent of the starting conditions. For certain values of the system parameters the oscillation is almost sinusoidal, while in a different range the oscillation shows abrupt changes, see fig. 1.1.1. In the last case the period turns out to be about proportional to a large parameter of the system. The name "relaxation oscillation" refers to this characteristic time constant of the system. Le Corbeiller (1931) took over this name. In fig. 1.1.2 the triode circuit of Van der Pol is given. The system satisfies the following equation

$$L\frac{dI}{dt} + RI + V = M\frac{dI_a}{dt}, \tag{1.1.1}$$

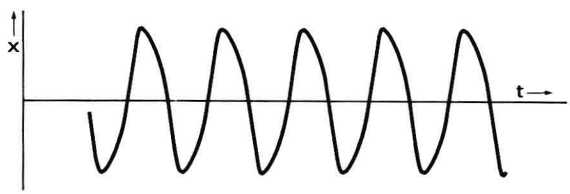
where L is the selfinductance, M the mutual inductance, R a resistance and I and V, respectively, a current and the grid voltage. Assuming that the grid current is negligible we have $I = CV'(t)$, where C is a capacitance. The triode characteristic is idealized as $I_a = V - 1/3V^3$. Then by substituting

$$V = x\sqrt{1 - RC/M}, \quad t = \tau\sqrt{LC} \tag{1.1.2ab}$$

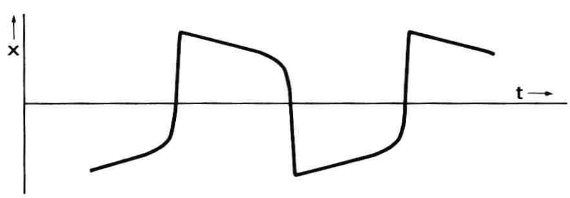
we obtain the well-known Van der Pol equation



(a) $\nu = 0.1$



(b) $\nu = 1.0$



(c) $\nu = 10.$

Fig.1.1.1 Periodic solution of the Van der Pol equation for different values of the parameter. For $\nu = 0.1$ the oscillation is almost sinusoidal with a period of about 2π, while for $\nu = 10.$ the oscillation is almost discontinuous with a period of about $(3 - 2\ln 2)\nu$.