ALGEBRAIC GRAPH THEORY

Second Edition

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Preface

This book is a substantially enlarged version of the Cambridge Tract with the same title published in 1974. There are two major changes.

- The main text has been thoroughly revised in order to clarify the exposition, and to bring the notation into line with current practice. In the course of revision it was a pleasant surprise to find that the original text remained a fairly good introduction to the subject, both in outline and in detail. For this reason I have resisted the temptation to reorganise the material in order to make the book rather more like a standard textbook.

- Many *Additional Results* are now included at the end of each chapter. These replace the rather patchy selection in the old version, and they are intended to cover most of the major advances in the last twenty years. It is hoped that the combination of the revised text and the additional results will render the book of service to a wide range of readers.

I am grateful to all those people who have helped by commenting upon the old version and the draft of the new one. Particular thanks are due to Peter Rowlinson, Tony Gardiner, Ian Anderson, Robin Wilson, and Graham Brightwell. On the practical side, I thank Alison Adcock, who prepared a TeX version of the old book, and David Tranah of Cambridge University Press, who has been constant in his support.

Norman Biggs March 1993

Contents

1

Introduction to algebraic graph theory

About the book

This book is concerned with the use of algebraic techniques in the study of graphs. The aim is to translate properties of graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorems about graphs.

It is fortunate that the basic terminology of graph theory has now become part of the vocabulary of most people who have a serious interest in studying mathematics at this level. A few basic definitions are gathered together at the end of this chapter for the sake of convenience and standardization. Brief explanations of other graph-theoretical terms are included as they are needed. A small number of concepts from matrix theory, permutation-group theory, and other areas of mathematics, are used, and these are also accompanied by a brief explanation.

The literature of algebraic graph theory itself has grown enormously since 1974, when the original version of this book was published. Literally thousands of research papers have appeared, and the most relevant ones are cited here, both in the main text and in the Additional Results at the end of each chapter. But no attempt has been made to provide a complete bibliography, partly because there are now several books dealing with aspects of this subject. In particular there are two books which contain massive quantities of information, and on which it is convenient to rely for 'amplification and exemplification' of the main results discussed here.

These are:

Spectra of Graphs: D.M. Cvetković, M. Doob, and H. Sachs, Academic Press (New York) 1980.

Distance-Regular Graphs: A.E. Brouwer, A.M. Cohen, and A. Neumaier, Springer-Verlag (Berlin) 1989.

References to these two books are given in the form [CvDS, p. 777], and [BCN, p. 888].

C.D. Godsil's recent book *Algebraic Combinatorics* (Chapman and Hall, 1993) arrived too late to be quoted as reference. It is in many ways complementary to this book, since it covers several of the same topics from a different point of view. Finally, the long-awaited *Handbook of Combinatorics* will contain authoritative accounts of many subjects discussed in these pages.

Outline of the book

The book is in three parts, each divided into a number of short chapters. The first part deals with the applications of linear algebra and matrix theory to the study of graphs. We begin by introducing the adjacency matrix of a graph; this matrix completely determines the graph, and its spectral properties are shown to be related to properties of the graph. For example, if a graph is regular, then the eigenvalues of its adjacency matrix are bounded in absolute value by the degree of the graph. In the case of a line graph, there is a strong lower bound for the eigenvalues. Another matrix which completely describes a graph is the incidence matrix of the graph. This matrix represents a linear mapping which determines the homology of the graph. The problem of choosing a basis for the homology of a graph is just that of finding a fundamental system of cycles, and this problem is solved by using a spanning tree. At the same time we study cuts in the graph. These ideas are then applied to the systematic solution of network equations, a topic which supplied the stimulus for the original theoretical development. We then investigate formulae for the number of spanning trees in a graph, and results which are derived from the expansion of determinants. These expansions illuminate the relationship between a graph and the characteristic polynomial of its adjacency matrix. The first part ends with a discussion of how spectral techniques can be used in problems involving partitions of the vertex-set, such as the vertex-colouring problem.

The second part of the book deals with the colouring problem from a different point of view. The algebraic technique for counting the colourings of a graph is founded on a polynomial known as the chromatic polynomial. We first discuss some simple ways of calculating this polynomial, and show how these can be applied in several important cases. Many important properties of the chromatic polynomial of a graph stem from its connection with the family of subgraphs of the graph, and we show how the chromatic polynomial can be expanded in terms of subgraphs. From the first (additive) expansion another (multiplicative) expansion can be derived, and the latter depends upon a very restricted class of subgraphs. This leads to efficient methods for approximating the chromatic polynomials of large graphs. A completely different kind of expansion relates the chromatic polynomial to the spanning trees of a graph; this expansion has several remarkable features and leads to new ways of looking at the colouring problems, and some new properties of chromatic polynomials.

The third part of the book is concerned with symmetry and regularity properties. A symmetry property of a graph is related to the existence of automorphisms – that is, permutations of the vertices which preserve adjacency. A regularity property is defined in purely numerical terms. Consequently, symmetry properties induce regularity properties, but the converse is not necessarily true. We first study the elementary properties of automorphisms, and explain the connection between the automorphisms of a graph and the eigenvalues of its adjacency matrix. We then introduce a hierarchy of symmetry conditions which can be imposed on a graph, and proceed to investigate their consequences. The condition that all vertices be alike (under the action of the group of automorphisms) turns out to be rather a weak one, but a slight strengthening of it leads to highly non-trivial conclusions. In fact, under certain conditions, there is an absolute bound to the level of symmetry which a graph can possess. A strong symmetry property, called distance-transitivity, and the consequent regularity property, called distance-regularity, are then introduced. We return to the methods of linear algebra to derive numerical constraints upon the existence of graphs with these properties. Finally, these constraints are applied to the problem of finding minimal regular graphs whose degree and girth are given.

Basic definitions and notation

Formally, a *general graph* Γ consists of three things: a set $V\Gamma$, a set $E\Gamma$, and an incidence relation, that is, a subset of $V\Gamma \times E\Gamma$. An element of $V\Gamma$ is called a *vertex*, an element of $E\Gamma$ is called an *edge*, and the incidence relation is required to be such that an edge is incident with either one vertex (in which case it is a *loop*) or two vertices. If every edge is incident with two vertices, and no two edges are incident with the same pair of vertices, then we say that Γ is a ***strict graph*** or briefly, a *graph*. In this case, $E\Gamma$ can be regarded as a subset of the set of unordered pairs of vertices. We shall deal mainly with graphs (that is, strict graphs), except in Part Two, where it is sometimes essential to consider general graphs.

If v and w are vertices of a graph Γ, and $e = \{v, w\}$ is an edge of Γ, then we say that e *joins* v and w, and that v and w are the *ends* of e. The number of edges of which v is an end is called the *degree* of v. A *subgraph* of Γ is constructed by taking a subset S of $E\Gamma$ together with all vertices incident in Γ with some edge belonging to S. An *induced subgraph* of Γ is obtained by taking a subset U of $V\Gamma$ together with all edges which are incident in Γ only with vertices belonging to U. In both cases the incidence relation in the subgraph is inherited from the incidence relation in Γ. We shall use the notation $\langle S\rangle_\Gamma$, $\langle U\rangle_\Gamma$ for these subgraphs, and usually, when the context is clear, the subscript Γ will be omitted.

PART ONE

Linear algebra in graph theory

2

The spectrum of a graph

We begin by defining a matrix which will play an important role in many parts of this book. Suppose that Γ is a graph whose vertex-set $V\Gamma$ is the set $\{v_1, v_2, ..., v_n\}$, and consider $E\Gamma$ as a set of unordered pairs of elements of $V\Gamma$. If $\{v_i, v_j\}$ is in $E\Gamma$, then we say that v_i and v_j are *adjacent*.

Definition 2.1 The *adjacency matrix* of Γ is the $n \times n$ matrix $\mathbf{A} = \mathbf{A}(\Gamma)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

For the sake of definiteness we consider $\mathbf{A}$ as a matrix over the complex field. Of course, it follows directly from the definition that $\mathbf{A}$ is a real symmetric matrix, and that the trace of $\mathbf{A}$ is zero. Since the rows and columns of $\mathbf{A}$ correspond to an arbitrary labelling of the vertices of Γ, it is clear that we shall be interested primarily in those properties of the adjacency matrix which are invariant under permutations of the rows and columns. Foremost among such properties are the spectral properties of $\mathbf{A}$.

Suppose that λ is an eigenvalue of $\mathbf{A}$. Then, since $\mathbf{A}$ is real and symmetric, it follows that λ is real, and the multiplicity of λ as a root of the equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ is equal to the dimension of the space of eigenvectors corresponding to λ.

Definition 2.2 The *spectrum* of a graph Γ is the set of numbers which are eigenvalues of $\mathbf{A}(\Gamma)$, together with their multiplicities. If the distinct eigenvalues of $\mathbf{A}(\Gamma)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$, and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write

$$\operatorname{Spec}\Gamma = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}.$$

For example, the *complete graph* K_n is the graph with n vertices in which each distinct pair are adjacent. Thus the graph K_4 has adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and an easy calculation shows that the spectrum of K_4 is

$$\operatorname{Spec} K_4 = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

We shall usually refer to the eigenvalues of $\mathbf{A} = \mathbf{A}(\Gamma)$ as the *eigenvalues of* Γ. Also, the characteristic polynomial $\det(\lambda\mathbf{I} - \mathbf{A})$ will be referred to as the *characteristic polynomial of* Γ, and denoted by $\chi(\Gamma;\lambda)$. Let us suppose that the characteristic polynomial of Γ is

$$\chi(\Gamma;\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \ldots + c_n.$$

In this form we know that $-c_1$ is the sum of the zeros, that is, the sum of the eigenvalues. This is also the trace of $\mathbf{A}$ which, as we have already noted, is zero. Thus $c_1 = 0$. More generally, it is proved in the theory of matrices that all the coefficients can be expressed in terms of the *principal minors* of $\mathbf{A}$, where a principal minor is the determinant of a submatrix obtained by taking a subset of the rows and the same subset of the columns. This leads to the following simple result.

Proposition 2.3 *The coefficients of the characteristic polynomial of a graph* Γ *satisfy:*

(1) $c_1 = 0$;

(2) $-c_2$ *is the number of edges of* Γ;

(3) $-c_3$ *is twice the number of triangles in* Γ.

Proof For each $i \in \{1, 2, ..., n\}$, the number $(-1)^i c_i$ is the sum of those principal minors of $\mathbf{A}$ which have i rows and columns. So we can argue as follows.

(1) Since the diagonal elements of $\mathbf{A}$ are all zero, $c_1 = 0$.

(2) A principal minor with two rows and columns, and which has a

non-zero entry, must be of the form

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

There is one such minor for each pair of adjacent vertices of Γ, and each has value -1. Hence $(-1)^2 c_2 = -|E\Gamma|$, giving the result.

(3) There are essentially three possibilities for non-trivial principal minors with three rows and columns:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix},$$

and, of these, the only non-zero one is the last (whose value is 2). This principal minor corresponds to three mutually adjacent vertices in Γ, and so we have the required description of c_3. □

These simple results indicate that the characteristic polynomial of a graph is an object of the kind we study in algebraic graph theory: it is an algebraic construction which contains graphical information. Proposition 2.3 is just a pointer, and we shall obtain a more comprehensive result on the coefficients of the characteristic polynomial in Chapter 7.

Suppose $\mathbf{A}$ is the adjacency matrix of a graph Γ. Then the set of polynomials in $\mathbf{A}$, with complex coefficients, forms an algebra under the usual matrix operations. This algebra has finite dimension as a complex vector space. Indeed, the Cayley–Hamilton theorem asserts that $\mathbf{A}$ satisfies its own characteristic equation, so the dimension is at most n, the number of vertices in Γ.

Definition 2.4 The *adjacency algebra* of a graph Γ is the algebra of polynomials in the adjacency matrix $\mathbf{A} = \mathbf{A}(\Gamma)$. We shall denote the adjacency algebra of Γ by $\mathcal{A}(\Gamma)$.

Since every element of the adjacency algebra is a linear combination of powers of $\mathbf{A}$, we can obtain results about $\mathcal{A}(\Gamma)$ from a study of these powers. We define a *walk* of length l in Γ, from v_i to v_j, to be a finite sequence of vertices of Γ,

$$v_i = u_0, u_1, \ldots, u_l = v_j,$$

such that u_{t-1} and u_t are adjacent for $1 \leq t \leq l$.

Lemma 2.5 *The number of walks of length l in Γ, from v_i to v_j, is the entry in position (i, j) of the matrix $\mathbf{A}^l$.*

Proof The result is true for $l = 0$ (since $\mathbf{A}^0 = \mathbf{I}$) and for $l = 1$ (since $\mathbf{A}^1 = \mathbf{A}$ is the adjacency matrix). Suppose that the result is true for $l = L$. The set of walks of length $L + 1$ from v_i to v_j is in bijective correspondence with the set of walks of length L from v_i to vertices v_h adjacent to v_j. Thus the number of such walks is

$$\sum_{\{v_h,v_j\}\in E\Gamma} (\mathbf{A}^L)_{ih} = \sum_{h=1}^{n} (\mathbf{A}^L)_{ih} a_{hj} = (\mathbf{A}^{L+1})_{ij}.$$

It follows that the number of walks of length $L+1$ joining v_i to v_j is $(\mathbf{A}^{L+1})_{ij}$. The general result follows by induction. □

A graph is said to be *connected* if each pair of vertices is joined by a walk. The number of edges traversed in the shortest walk joining v_i and v_j is called the *distance* in Γ between v_i and v_j and is denoted by $\partial(v_i, v_j)$. The maximum value of the distance function in a connected graph Γ is called the *diameter* of Γ.

Proposition 2.6 *Let Γ be a connected graph with adjacency algebra $\mathcal{A}(\Gamma)$ and diameter d. Then the dimension of $\mathcal{A}(\Gamma)$ is at least $d+1$.*

Proof Let x and y be vertices of Γ such that $\partial(x, y) = d$, and suppose that

$$x = w_0, w_1, \ldots, w_d = y$$

is a walk of length d. Then, for each $i \in \{1, 2, \ldots, d\}$, there is at least one walk of length i, but no shorter walk, joining w_0 to w_i. Consequently, $\mathbf{A}^i$ has a non-zero entry in a position where the corresponding entries of $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^{i-1}$ are zero. It follows that $\mathbf{A}^i$ is not linearly dependent on $\{\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^{i-1}\}$, and that $\{\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^d\}$ is a linearly independent set in $\mathcal{A}(\Gamma)$. Since this set has $d+1$ members, the proposition is proved. □

There is a close connection between the adjacency algebra and the spectrum of Γ. If the adjacency matrix has s distinct eigenvalues then, since it is a real symmetric matrix, its minimum polynomial (the monic polynomial of least degree which annihilates it) has degree s. Consequently the dimension of the adjacency algebra is equal to s. Thus we have the following bound for the number of distinct eigenvalues.

Corollary 2.7 *A connected graph with diameter d has at least $d+1$ distinct eigenvalues.* □

One of the major topics of the last part of this book is the study of a class of 'highly regular' connected graphs which have the minimum number $d+1$ of distinct eigenvalues. In the following chapters we shall encounter several other examples of the link between structural regularity and the spectrum.

Notation The eigenvalues of a graph may be be listed in two ways: in strictly decreasing order of the distinct values, as in Definition 2.2, or in weakly decreasing order (with repeated values) $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$, where $n = |V\Gamma|$. We shall use either method, as appropriate.

Additional Results

2a *A reduction formula for* χ Suppose Γ is a graph with a vertex v_1 of degree 1, and let v_2 be the vertex adjacent to v_1. Let Γ_1 be the induced subgraph obtained by removing v_1, and Γ_{12} the induced subgraph obtained by removing $\{v_1, v_2\}$. Then

$$\chi(\Gamma;\lambda) = \lambda\chi(\Gamma_1;\lambda) - \chi(\Gamma_{12};\lambda).$$

This formula can be used to calculate the characteristic polynomial of any tree, because a tree always has a vertex of degree 1. A more general reduction formula was found by Rowlinson (1987).

2b *The characteristic polynomial of a path* Let P_n be the *path graph* with vertex-set $\{v_1, v_2, \ldots, v_n\}$ and edges $\{v_i, v_{i+1}\}$ $(1 \leq i \leq n-1)$. For $n \geq 3$ we have

$$\chi(P_n;\lambda) = \lambda\chi(P_{n-1};\lambda) - \chi(P_{n-2};\lambda).$$

Hence $\chi(P_n;\lambda) = U_n(\lambda/2)$, where U_n denotes the *Chebyshev polynomial of the second kind.*

2c *The spectrum of a bipartite graph* A graph is *bipartite* if its vertex-set can be partitioned into two parts V_1 and V_2 such that each edge has one vertex in V_1 and one vertex in V_2. If we order the vertices so that those in V_1 come first, then the adjacency matrix of a bipartite graph takes the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^t & \mathbf{0} \end{bmatrix}.$$

If $\mathbf{x}$ is an eigenvector corresponding to the eigenvalue λ, and $\widetilde{\mathbf{x}}$ is obtained from $\mathbf{x}$ by changing the signs of the entries corresponding to vertices in V_2, then $\widetilde{\mathbf{x}}$ is an eigenvector corresponding to the eigenvalue $-\lambda$. It follows that the spectrum of a bipartite graph is symmetric with respect to 0, a result originally obtained by Coulson and Rushbrooke (1940) in the context of theoretical chemistry.

2d *The derivative of* χ For $i = 1, 2, \ldots, n$ let Γ_i denote the induced subgraph $\langle V\Gamma \setminus v_i\rangle$. Then

$$\chi'(\Gamma;\lambda) = \sum_{i=1}^{n} \chi(\Gamma_i;\lambda).$$