Probability and
Its Applications

R.K. Getoor

Excessive Measures

Birkhäuser

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Birkhäuser
Boston • Basel • Berlin

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Library of Congress Cataloging-in-Publication Data
Getoor, R.K. (Ronald Kay), 1929–
Excessive measures/R.K. Getoor.
p. cm.—(Probability and its applications)
Includes bibliographical references.
ISBN 0-8176-3492-4 (alk. paper)
1. Excessive measures (Mathematics) 2. Markov processes.
3. Measure theory. 4. Potential theory (Mathematics) I. Title.
II. Series.
QA274.7.G47 1990 89-18654
519.2′33—dc20

Printed on acid-free paper.

ISBN 0-8176-3492-4
ISBN 3-7643-3492-4

Camera-ready copy supplied by author using TEX.
Printed and bound by Edwards Brothers, Inc., Ann Arbor, Michigan.
Printed in the U.S.A.

9 8 7 6 5 4 3 2 1

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Preface

The study of the cone of excessive measures associated with a Markov process goes back to Hunt's fundamental memoir [**H57**]. However until quite recently it received much less attention than the cone of excessive functions. The fact that an excessive function can be composed with the underlying Markov process to give a supermartingale, subject to secondary finiteness hypotheses, is crucial in the study of excessive functions. The lack of an analogous construct for excessive measures seemed to make them much less tractable to a probabilistic analysis. This point of view changed radically with the appearance of the pioneering paper by Fitzsimmons and Maisonneuve [**FM86**] who showed that a certain stationary process associated with an excessive measure could be used to study excessive measures probabilistically. These stationary processes or measures had been constructed by Kuznetsov [**Ku74**] extending earlier work of Dynkin. It is now common to call them Kuznetsov measures. Following the Fitzsimmons-Maisonneuve paper there was renewed interest and remarkable progress in the study of excessive measures. The purpose of this monograph is to organize under one cover and prove under standard hypotheses many of these recent results in the theory of excessive measures.

The two basic tools in this recent development are Kuznetsov measures mentioned above and the energy functional. The energy functional has a long history that may be traced back to Hunt, but its systematic use in the study of excessive measures seems to be more recent. However, see [**CL75**] for its definition and use in an abstract setting. Also it was used for other purposes by Meyer in [**Me68**] and [**Me73**]. A third ingredient in this development is the use of two Riesz type decompositions of an excessive measure: the first into dissipative and conservative parts is due to Dynkin [**Dy80**]—see also Blumenthal [**B1**]; the second into a potential and a harmonic part is due originally to Getoor and Glover [**GG84**]. Both of these decompositions, as well as the more elementary decomposition into purely excessive and invariant parts, were given probabilistic interpretations in terms of Kuznetsov measures in [**FM86**].

Using these tools one can construct a potential theory for excessive measures that in many respects is closer to classical potential theory than the potential theory of excessive functions. In classical potential theory or, more generally, under strong duality assumptions as in Chapter VI of [**BG**], there is an isomorphism between excessive measures and (a class of) coexcessive functions. It turns out that many of the fine results about excessive functions under these hypotheses, when interpreted as theorems about coexcessive measures, have a natural extension to a general Markov process, even though the corresponding results for excessive functions do not generalize completely. Thus the natural generalization of certain classical results only appears in the potential theory of excessive measures. Of course, there exist generalizations of the classical theory to abstract cones that include both excessive measures and functions. However, our emphasis here is on the underlying probabilistic meaning of the potential theory.

One other important benefit of this approach is that it requires no a priori transience hypothesis: the transience assumptions being subsumed under the conservative-dissipative dichotomy. It is remarkable that often the results are the same in the two cases, although the proofs may be quite different.

In the first five sections we develop the theory of excessive measures about as far as we can without using Kuznetsov measures. In sections 6 and 7 we introduce Kuznetsov measures and use them to study excessive measures and their potential theory. These first seven sections contain the basic potential theory of excessive measures. Sections 8, 10 and 11 contain other important applications of the energy functional and Kuznetsov measures, but the role of excessive measures is somewhat secondary. Section 9 on flows and Palm measures is perhaps tangential to the main development, but is important for a better understanding of sections 8 and 10. Appendix A contains an expanded proof of Meyer's perfection theory for multiplicative functionals [**Me74**]. Although one could avoid it—a cost, of course—I have decided to include it because of its importance. It is often quoted and is deserving of an expanded proof.

My guiding principle during the writing was to give complete proofs of all results that are not available in the standard reference books listed at the beginning of the bibliography. Like most principles, it is easier to formulate in the abstract than follow in the particular. One consequence of this is that I refer to these standard books for needed facts whenever possible, rather than to the original papers in which they appeared. (As of this writing only a preliminary version of Chapters XVII and XVIII of the final part of the monumental treatise [**DM**] by Dellacherie and Meyer is available to me, and so references to these two chapters may not be completely accurate when the definitive version appears.)

It is a pleasure to acknowledge a few of my debts. First and most of all I must thank Pat Fitzsimmons. Even the casual reader will notice the extent to which his ideas pervade this work. But I owe him much more. I had the privilege of consulting with him on an almost daily basis during the writing of this volume. Time and time again he set me straight and pointed the way when I was stuck or confused. It seems unlikely that this work would have been completed without his help. I would especially like to thank him for his contributions to Appendix A and to acknowledge that the concept of a "good partition" used there is due to him. Last but not least he read the entire manuscript and made numerous suggestions for improving the exposition. Jutta Steffens supplied the key step in the proof of Proposition 4.17. This enabled me to sim-

plify considerably the original proof of the important Theorem 7.9 which gives the probabilistic meaning of Hunt's balayage operation on excessive measures. Neola Crimmins displayed her customary superb skill as well as unlimited patience in transforming my handwritten scrawl into the beautiful TEX format. Finally I received support from the National Science Foundation under NSF Grant DMS87-21347 during part of the writing.

La Jolla, California
September, 1989

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Contents

1. Notation and Preliminaries

We shall assume once and for all that

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$$

is a right Markov process as defined in §8 of [**S**] with state space $(E, \mathcal{E})$, semigroup (P_t), and resolvent (U^q). To be explicit E is a separable Radon space and $\mathcal{E}$ is the Borel σ-algebra of E. A cemetery point Δ is adjoined to E as an isolated point and $E_\Delta := E \cup \{\Delta\}$, $\mathcal{E}_\Delta := \sigma(\mathcal{E} \cup \{\Delta\})$. (The symbol "$:=$" should be read as "is defined to be".) We suppose that [**S**, (20.5)] holds; that is, $X_t(\omega) = \Delta$ implies that $X_s(\omega) = \Delta$ for all $s \geq t$ and that there is a point $[\Delta]$ in Ω (the dead path) with $X_t([\Delta]) = \Delta$ for all $t \geq 0$. Of course, $\zeta = \inf\{t: X_t = \Delta\}$ is the lifetime of X. The filtration $(\mathcal{F}, \mathcal{F}_t)$ is the augmented natural filtration of X, [**S**, (3.3)]. We shall always use $\mathcal{E}$ to denote the Borel σ-algebra of E in the original topology of E. Beginning in §20, Sharpe uses $\mathcal{E}$ to denote the Borel σ-algebra of E in the Ray topology. We shall *not* use this convention. We shall write $\mathcal{E}^r$ for the σ-algebra of Ray Borel sets. These assumptions on X are weaker than those in [**G**] or [**DM**, XVI-4]. Beginning in §6 we shall make an additional assumption on X. (See (6.2)). To avoid trivialities *we assume throughout this monograph that* $X_\infty(\omega) = \Delta$, $\theta_0\omega = \omega$, *and* $\theta_\infty\omega = [\Delta]$ *for all* $\omega \in \Omega$.

Our notation for the objects associated with X is the standard notation (with few exceptions) that may be found in the familiar reference books [**BG**], [**DM**], [**G**], and [**S**]. For the convenience of the reader we shall recall the basic ones as well as some of their familiar properties. We refer the reader to the above references for proofs.

If $(H, \mathcal{H})$ is a measurable space, $\mathcal{H}^*$ denotes the σ-algebra of universally measurable sets over $(H, \mathcal{H})$. If $\mathcal{C}$ is any collection of extended real valued functions on H, then $p\mathcal{C}$ (resp. $b\mathcal{C}$) denotes those $f \in \mathcal{C}$ which are positive (resp. bounded). In particular $p\mathcal{H}$ (resp. $b\mathcal{H}$) denotes the collection of positive (resp. bounded) $\mathcal{H}$ measurable functions on H.

S^q or $S^q(X)$ denotes the cone of q-excessive functions of X, $q \geq 0$. As usual we write $S = S^0$. Perhaps the most useful σ-algebra on E is $\mathcal{E}^e := \sigma\left(\bigcup_{q \geq 0} S^q\right)$. It is immediate from the resolvent equation that $\mathcal{E}^e = \sigma(S^q)$ for any $q > 0$. Also P_t and U^q map $\mathcal{E}^e$ into itself. The potential kernel of X, $U := U^0$ is *proper* provided there exists $f \in \mathcal{E}^*$ with $f > 0$ and $Uf < \infty$. Then there exists $g \in \mathcal{E}^e$ with $g > 0$ and $Ug \leq 1$. See [**DM**, XII-8] or [**G80**]. In fact one may even suppose that g is finely continuous. If U is proper and $u \in S$, then there exists a sequence $(f_n) \subset pb\mathcal{E}^e$ with $Uf_n \uparrow u$ [**DM**, XII-17]. The process X is *transient* provided U is proper. A function $f \in p\mathcal{E}^*$ is *supermedian* provided $P_t f \leq f$ for each $t \geq 0$. Then $\hat{f} := \uparrow \lim_{t \downarrow 0} P_t f$ is excessive and $\hat{f}$ is called the *excessive regularization* of f. We write $P_t^q = e^{-qt} P_t$ for $q > 0$. This is the semigroup of X^q, the q-subproces of X. Clearly $S^q(X) = S(X^q)$. Since X^q is a right process all of the above considerations may be applied to X^q. In particular, X^q is transient when $q > 0$. We let $\mathcal{F}_t^e$ (resp. $\mathcal{F}^e$) denote the σ-algebra generated by $f \circ X_s$ with $f \in \mathcal{E}^e$ and $s \leq t$ (resp. $s < \infty$).

A σ-finite measure ξ on $(E, \mathcal{E})$ is q-*excessive* provided $\xi P_t^q \leq \xi$ for each $t \geq 0$. From now on we shall just say ξ is a measure on E when we mean a measure on $(E, \mathcal{E})$. Of course, any measure ξ on E has a unique extension to $\mathcal{E}^*$ which we again denote by ξ. Let Exc^q or $\mathrm{Exc}^q(X)$ denote the class of q-excessive measures. We drop q from the notation when it has the value zero. Thus Exc denotes the class of excessive measures. Obviously $\mathrm{Exc}^q(X) = \mathrm{Exc}(X^q)$. If $\xi \in \mathrm{Exc}^q$, then

$\xi P_t^q \uparrow \xi$ as $t \downarrow 0$ [**DM**, XII-37b]. Suppose $\xi \in \text{Exc}$ and U is proper. Then there exists a sequence of finite measures (μ_n) on E with $\mu_n U \uparrow \xi$ [**DM**, XII-38]. Clearly if $\xi, \eta \in \text{Exc}$, then $\xi \wedge \eta \in \text{Exc}$ and if (ξ_n) is an increasing sequence of excessive measures, then $\xi := \uparrow \lim \xi_n \in \text{Exc}$ provided ξ is σ-finite.

If $B \in \mathcal{E}^e$, then $T_B := \inf\{t > 0\colon X_t \in B\}$ where the infimum of the empty set is $+\infty$ is an $(\mathcal{F}_t)$ stopping time. If $B \in \mathcal{E}$, then T_B is even an $(\mathcal{F}^*_{t+})$ stopping time; that is $\{T_B < t\} \in \mathcal{F}^*_t$ for each $t > 0$. Here $\mathcal{F}^*_t$ is the universal completion of $\mathcal{F}^0_t := \sigma(X_s\colon s \le t)$ and $\mathcal{F}^*_t \subset \mathcal{F}_t$ for each t. T_B is the *hitting time* of B. Define for $f \in p\mathcal{E}^*$

$$(1.1)\quad P_B^q f(x) = P^x[e^{-qT_B} f \circ X_{T_B}] := \int e^{-qT_B} f \circ X_{T_B}\, dP^x .$$

where, by convention, $e^{-0T_B} = 1_{\{T_B<\infty\}}$. We also use the convention that any function f on E is extended to E_Δ by $f(\Delta) = 0$. Thus in (1.1) the integration in ω is only over the set $\{T_B < \zeta\}$. Clearly P_B^q is a kernel on $(E, \mathcal{E}^*)$. If μ is a finite measure on E and $B \in \mathcal{E}^e$, then there exists an increasing sequence (K_n) of compact subsets of B with $T_{K_n} \downarrow T_B$ a.s. P^μ [**G**, (12.15)]. If $B \in \mathcal{E}^e$, B^r denotes the set of regular points for B, $B^r := \{x\colon P^x(T_B = 0) = 1\}$ and $B^r \in \mathcal{E}^e$. It is important to note that B^r and P_B^q depend only on the semigroup (P_t) and not on the particular realization, X, of (P_t) as a right process. See §19 of [**S**]; especially (19.7) in which nearly optional should be replaced by $\mathcal{E}^e$ measurable. (The proof of (19.7ii) as stated in [**S**] is incomplete.)

The definition of the fine topology in [**S**] should be modified slightly. The definition (10.7) in [**S**] should be changed to read as follows: A subset G of E is *finely open* provided for each $x \in G$ there exists $B \in \mathcal{E}^e$ such that $x \in B \subset G$ and $x \notin (B^c)^r$. (Here $B^c := E \backslash B$ and our B corresponds to B^c in [**S**, (10.7)]. The critical difference is that we require $B \in \mathcal{E}^e$ rather than nearly optional.) By the zero-one law $x \notin (B^c)^r$ if and only if $P^x(T_{B^c} > 0) = 1$. The fine topology on E is the collection of finely open sets. With this definition the fine topology depends only on the semigroup (P_t) and not on the particular realization, X, of (P_t) as a right process. Consequently in proving results about the fine topology one may suppose X satisfies the conditions in §20 of [S] without loss of generality. With this modification in the definition of the fine topology, the proof of Theorem 49.9 in [S] is valid. It states that the fine topology is generated by $\bigcup_{q\geq 0} S^q$, or just by S^q for any $q > 0$. The proof of the following proposition is a straightforward adaptation of the proof of [**BG**, II-(4.6)].

(1.2) **Proposition.** *Suppose U is proper. Then the fine topology is generated by S.*

Proof. As observed earlier we may suppose X satisfies the conditions in §20 of [S] in the proof. Let $\mathcal{T}$ be the topology generated by S and let $\mathcal{O}(f)$ be the fine topology. Then it suffices to show $\mathcal{O}(f) \subset \mathcal{T}$. The proof of [S, (49.9)] shows that sets of the form $\{\psi_G < 1\}$ where $\psi_G := P^{\cdot}(e^{-T_G})$ as G ranges over all open sets in the Ray topology of E form a base for $\mathcal{O}(f)$. Fix such a G and let $\psi = \psi_G$. Since U is proper, there exists $0 < h \leq 1$ with $h \in \mathcal{E}^e$ and $Uh \leq 1$. Let $f_n = (nh) \wedge 1$. Then $f_n > 0$, $f_n \uparrow 1$, and $Uf_n \leq n$. Hence

$$\varphi_n := Uf_n - P_G U f_n = P^{\cdot} \int_0^{T_G} f_n \circ X_t \, dt \uparrow P^{\cdot}(T_G \wedge \zeta).$$

Consequently $\varphi := P^{\cdot}(T_G \wedge \zeta)$ is $\mathcal{T}$-l.s.c. If $\varphi(x) > 0$, then $P^x(T_G > 0) = 1$ and so $\psi(x) < 1$. But $P^x(T_G > 0) = 1$ if $\psi(x) < 1$ and thus $\varphi(x) > 0$. Therefore $\{\psi < 1\} = \{\varphi > 0\} \in \mathcal{T}$. Hence $\mathcal{O}(f) \subset \mathcal{T}$. ∎

We close this section with some additional notational conventions which we shall use without special mention in the sequel. Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be measurable spaces. We write $f \in \mathcal{M}|\mathcal{N}$ to indicate that f is a measurable mapping from M to N; that is, $f: M \to N$ and $f^{-1}(\mathcal{N}) \subset \mathcal{M}$. If B is any subset of M, $\mathcal{M}|_B$ denotes the trace of $\mathcal{M}$ on B. If f is a numerical function on M, $\|f\| := \sup\{|f(x)|: x \in M\}$.

If μ is a measure on $(M, \mathcal{M})$ and $f \in p\mathcal{M}^*$ we write $f\mu$ or $f \cdot \mu$ for the measure $B \to \int_B f\,d\mu$ defined on $(M, \mathcal{M})$. We also write $\mu(f)$ or $\langle \mu, f \rangle$ for $\int f\,d\mu$, and sometimes just μf. Thus $\mu Uf = \mu(Uf) = \mu U(f)$. All named functions on E — the state space of X —are in $p\mathcal{E}^*$ and all named subsets of E are in $\mathcal{E}^e$ unless explicitly stated otherwise. We use the American convention for limits. For example, if f is a numerical function on $\mathbb{R}$, we write $\lim_{s \downarrow t} f(s)$ rather than $\lim_{s \downarrow\downarrow t} f(s)$. However, we use the notation $t_n \downarrow\downarrow t$ to denote a sequence (t_n) with $t_n > t$ for each n and which decreases to t. We write $\uparrow \lim_{s \downarrow t} f(s) = L$ to indicate that $f(s)$ increases to L as s decreases to t, etc. We use the standard notation $\mathbb{Q}$ and $\mathbb{R}$ for the rationals and the reals respectively, while $\mathbb{Q}^+$, $\mathbb{R}^+$, and $\mathbb{R}^{++}$ denote $\mathbb{Q} \cap [0, \infty[$, $\mathbb{R} \cap [0, \infty[$ and $\mathbb{R} \cap]0, \infty[$ respectively. $\mathcal{B}$, $\mathcal{B}^+$, and $\mathcal{B}^{++}$ denote the Borel σ-algebras of $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^{++}$ respectively. As usual in the theory of Markov processes almost surely (a.s.) means a.s. P^μ for each initial measure μ on E. Similarly indistinguishable means P^μ indistinguishable for each μ.

There is one technical convention that we adopt deserving special mention. If $f \in p\mathcal{E}^*$, then one readily checks that $(t, \omega) \to f \circ X_t(\omega)$ is $(\mathcal{B}^+ \otimes \mathcal{F}^0)^*$ measurable. In particular if P is *any* probability on Ω, $t \to f \circ X_t(\omega)$ is Lebesgue measurable for P a.e. ω. Consequently integrals of the form $\int \varphi(t) f \circ X_t\,dt$ for $\varphi \in p\mathcal{B}^+$ are defined P a.e., and since P is arbitrary they are $\mathcal{F}^*$ measurable provided they are defined arbitrarily (or left undefined) on the set of ω's such that $t \to f \circ X_t(\omega)$ is not Lebesgue measurable. If we want or need an unambiguously defined function of ω we replace the integral by the outer integral, $\int^* \varphi(t) f \circ X_t(\omega)\,dt$, discussed in Appendix B. The reader should consult Appendix B for more details. In the sequel we shall use the symbol $\int \varphi(t) f \circ X_t\,dt$ in this, perhaps, somewhat ambiguous manner.

2. Excessive Measures

In this section we shall define several important subclasses of excessive measures and detail two Riesz type decompositions of excessive measures.

A σ-finite measure ξ on E is *invariant* provided $\xi = \xi P_t$ for each $t > 0$. We write Inv for the class of invariant measures. Clearly Inv $\subset$ Exc. We write Inv(X) or Inv(P_t) if we want to emphasize the process X or the semigroup (P_t). This same notational scheme will be used for the other classes of excessive measures to be introduced in this section. If μ is a measure on E, then it is easily checked that $(\mu U)P_t \leq \mu U$ and so $\mu U \in$ Exc if and only if it is σ-finite. Excessive measures of the form μU are called *potentials* and we write Pot for the class of potentials. If $f > 0$ then $Uf > 0$, and so if $\mu U \in$ Pot then μ must be σ-finite. The converse is false even if U is proper; for example, consider Brownian motion in three or more dimensions and take μ to be Lebesgue measure. On the other hand if U is proper and μ is a finite measure, then $\mu U \in$ Pot.

Suppose $\mu U \in$ Pot and $f \geq 0$ with $\mu U(f) < \infty$. Then

$$\mu U P_t(f) = \int_t^\infty \mu P_s(f)\, ds \to 0 \quad \text{as} \quad t \to \infty .$$

This leads to the following definition. An excessive measure ξ is *purely excessive* provided $\xi P_t(f) \to 0$ as $t \to \infty$ whenever $f \geq 0$ with $\xi(f) < \infty$. The class of purely excessive measures is denoted by Pur. Then Pot $\subset$ Pur $\subset$ Exc.

We need some elementary facts about measures. The standard proofs are left to the reader. If μ and ν are measures on E we write $\mu \leq \nu$ provided $\mu(B) \leq \nu(B)$ for all $B \in \mathcal{E}$ and, hence all $B \in \mathcal{E}^*$. If μ and ν are σ-finite measures and $\mu \le \nu$, then there exists a unique σ-finite measure, λ, such that $\mu + \lambda = \nu$. We write $\lambda = \nu - \mu$. If (μ_n) is a decreasing sequence of σ-finite measures, then there exists a unique σ-finite measure μ such that $\mu_n(B) \downarrow \mu(B)$ whenever $\mu_n(B) < \infty$ for some n. Then $\mu_n(f) \downarrow \mu(f)$ if $f \ge 0$ and $\mu_n(f) < \infty$ for some n. We write $\mu = \downarrow \lim \mu_n$ or simply $\mu_n \downarrow \mu$. This extends to an indexed family $(\mu_t)_{t>0}$ of σ-finite measures which is decreasing ($\mu_s \ge \mu_t$ if $s \le t$). Note that if (μ_n) (or (μ_t)) is decreasing and $\mu_n(f) \to 0$ for a single $f > 0$ with $\mu_n(f) < \infty$ for some n, then $\mu_n \downarrow 0$.

We come now to the first decomposition of excessive measures.

(2.1) **Theorem.** *Let $\xi \in \mathrm{Exc}$. Then ξ may be written uniquely as $\xi = \xi_i + \xi_p$ where $\xi_i \in \mathrm{Inv}$ and $\xi_p \in \mathrm{Pur}$. Moreover $\xi_i = \downarrow \lim_{t\to\infty} \xi P_t$.*

Proof. Since $\xi \in \mathrm{Exc}$, $(\xi P_t)_{t>0}$ is decreasing and so $\xi_i := \lim_{t\to\infty} \xi P_t$ exists as a σ-finite measure and $\xi_i \le \xi$. If $f \ge 0$ with $\xi(f) < \infty$ and $s > 0$, then $\xi_i(P_s f) = \downarrow \lim_{t\to\infty} \xi P_t(P_s f) = \xi_i(f)$. Consequently $\xi_i \in \mathrm{Inv}$. Now defining $\xi_p := \xi - \xi_i$, it follows from the invariance of ξ_i that $\xi_p \in \mathrm{Exc}$ and, hence, $\xi_p \in \mathrm{Pur}$. If $\xi = \eta + \lambda$ with $\eta \in \mathrm{Inv}$ and $\lambda \in \mathrm{Pur}$, then since $\lambda P_t \downarrow 0$ as $t \to \infty$ it follows that $\xi_i = \eta$, and then that $\xi_p = \lambda$. ■

The next decomposition is considerably less elementary than the one detailed in Theorem 2.1. It is due to Dynkin [**DY80**]. However, we shall follow Blumenthal [**B86**] in our discussion. We begin with an elementary fact.

(2.2) **Proposition.** *Let $\xi \in \mathrm{Pur}$. Then there exists an increasing sequence of potentials $(\mu_n U)$ with $\mu_n U \uparrow \xi$.*

Proof. Since $\xi \in \mathrm{Exc}$ we may define σ-finite measures $\mu_n := n[\xi - P_{1/n}\xi]$. Let $f \ge 0$ with $\xi(f) < \infty$. Then