Introduction to DIFFERENTIAL EQUATIONS with Boundary Value Problems

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Introduction to DIFFERENTIAL EQUATIONS with Boundary Value Problems

Student:

To help you make the most of your study time and improve your grades, we have developed the following supplement designed to accompany Andrews/Introduction to Differential Equations with Boundary Value Problems:

Student's Solutions Manual by Andrews/DeVoe,
ISBN 0-06-500002-1

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Preface

This book is intended as a one-semester or two-semester introductory treatment of differential equations and their applications. It is designed for students in mathematics, engineering, or science who have successfully completed the basic sequence of courses in calculus.

Because of the importance of differential equations in a variety of engineering and science areas, there are a number of applications to problems in the physical sciences, as well as some in the social and life sciences, that are prominent throughout the text. However, I have tried to avoid the temptation of introducing a multitude of applications in many diverse areas of application. My experience is that students are often distracted by too many different types of applications. A few basic applications readily make the point about the important role of differential equations in the real world. Moreover, I have kept the discussions involving applications at an elementary level so that a minimal background in the various sciences is required of the student or instructor to follow or understand them. In addition, I have tried to maintain a close relationship between mathematical theories and applications, whenever possible, by providing physical interpretations of some of the mathematical results.

The text contains the standard material that is found in the majority of introductory texts on differential equations, but contains some distinctive features that we list below.

Distinctive Features

- **Linear first-order equations:** A separate discussion of the notions of homogeneous and particular solutions in addition to the standard integrating factor technique for solving first-order linear equations (Chapter 2). I believe this provides a unifying point of view that is useful in later chapters dealing with linear equations of higher order.
- **Green's function:** Another novel feature of the text is the introduction of the causal Green's function (Chapter 5) for handling nonhomogeneous initial value problems in a systematic and physically meaningful fashion. This function is linked in Chapter 6 to the impulse response function which is basic in engineering applications involving the analysis of linear systems.
- **Qualitative methods:** Also included is a brief discussion of the qualitative methods used in oscillation theory (Chapter 5) and the stability of solutions of nonlinear systems of equations (Chapter 9). These brief exposures to qualitative methods permit the student to see how the general behavior of solutions to certain differential equations can still be determined in the absence of an explicit solution function.

- **Worked examples:** There are over 220 numbered worked examples, each of which is generally indicative of typical problems to be found in the exercise sets.
- **Exercises:** Nearly 1800 problems are included in the exercise sets, containing a blend of drill-like problems, some more difficult and some that extend the theory and applications beyond that discussed in the exposition. Problems that are considered more challenging are marked by a star (*).

Each chapter in the text begins with an overview of the topics to be covered in that chapter. In addition, each chapter contains a chapter summary in which the most important chapter topics are highlighted. The first eight chapters, which make up the bulk of material covered in most one-semester courses, also contain a review exercise set at the end of the chapter summary. Many of the chapters after the first four chapters are independent of each other so that various arrangements of topics can be made to suit individual course needs. Also, sections that can easily be omitted for a shorter course are marked [0].

Answers to all odd-numbered problems are provided at the end of the text to aid the students, while an instructor's Answer Key is available which contains answers to all problems (both odd and even-numbered problems). In addition, there is an accompanying Solutions Manual which contains detailed solutions to every other odd problem plus those even-numbered problems marked by a rectangular black bullet ■. Not included in the Solutions Manual are the review exercises at the end of the first eight chapters.

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L. C. Andrews

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[†] [0] indicates an optional section.

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CHAPTER 1

Basic Concepts

Differential equations play a fundamental role in engineering, mathematical and physical sciences, and the life sciences because they can be used in the formulation of many physical laws and relations. The development of the theory of differential equations is closely interlaced with the development of mathematics in general, and it is indeed difficult to separate the two. In fact, most of the famous mathematicians from the time of Newton and Leibniz had some part in the cultivation of this fascinating subject.

In a systematic development of the general theory of differential equations it is helpful to organize the various types of equations into **classes.** The reason is that all differential equations belonging to a particular class can often be solved by the same method. Therefore, in Section 1.2 we start with a discussion of some of the various classification schemes, emphasizing **order** and **linearity.**

In Section 1.3 we define what we mean by a **solution** of a differential equation. Here we discover that differential equations are peculiar in that they generally possess many different solutions, and for this reason we usually seek a specific function, called a **general solution,** with the property that all solutions can be obtained from it.

We introduce the notions of **initial value problem** and **boundary value problem** in Section 1.4. These are the names attached to those problems occurring in applications wherein the solution of a differential equation must satisfy certain additional **auxiliary conditions.** The study of these problems is so important in applications that we have devoted a significant portion of the text to developing the theory associated with them. In the last section of the chapter are some **historical comments** concerning differential equations.

1.1 Introduction

The theory of differential equations (DEs) has played an important role in science and engineering since the invention of the calculus by Newton† and Leibniz.‡ Problems in the physical sciences have subsequently been investigated primarily by formulating them as DEs. The first problems studied from this point of view came mostly from the field of mechanics.

The role of DEs in solving problems in mechanics is nicely illustrated by considering **Newton's second law** of motion. In beginning physics courses this law is commonly introduced by the simple algebraic formula

$$F = ma$$

For a single "particle" in motion, F denotes the force acting on the particle (which may be simply its weight), m is the mass of the particle (generally assumed constant), and a is its *acceleration*. In solving problems in mechanics, however, we are usually concerned also with the *velocity* and *position* of the particle as a function of time, not just with its acceleration. We may recall from calculus that acceleration is the time derivative of velocity $v = v(t)$, that is, $a = dv/dt$. Hence, Newton's second law can also be expressed by the equation

$$F = m\frac{dv}{dt}$$

Moreover, if $y = y(t)$ is the position of the particle at time t, then its velocity is related by $v = dy/dt$. Using this relation, we see that $dv/dt = d^2y/dt^2$, and Newton's second law now assumes the additional form

$$F = m\frac{d^2y}{dt^2}$$

Because they involve derivatives of unknown quantities, these last two equations are examples of DEs. Clearly, DEs can evolve quite naturally in the formulation of even rather simple problems in physics. Of course, DEs are now used extensively in all areas of physics and engineering, and more recently have also found their way into the social and life sciences.

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† Sir Isaac Newton (1642–1727) was born on Christmas Day in the countryside of England. Along with Leibniz, he is credited with the invention of the calculus (see also Section 1.6).

‡ Gottfried Wilhelm von Leibniz (1646–1716) was born in Leipzig. Known as both a philosopher and mathematician, he is credited with building a remarkable system of modern philosophy and being co-developer of the calculus (see also Section 1.6).

1.2 Classification of DEs

By a **differential equation** we mean simply an equation that is composed of a single unknown function and a finite number of its derivatives. One of the simplest examples that occurs early in the calculus is to find all functions y for which

$$y' = f(x) \tag{1}$$

where $f(x)$ is a given function. For instance, if $f(x) = x^2$, the unknown function y is obtained through a simple integration to yield

$$y = \frac{x^3}{3} + C \tag{2}$$

where C is a constant of the integration which can assume any value.

Most of the DEs that concern us in this text are not of the simple variety as described by Equation (1). Typical examples, some of which we discuss in later chapters, include the following:

$$y' = x^2 y^3 \tag{3}$$

$$y'' + k^2 y = \sin x \tag{4}$$

$$y'' + b \sin y = 0 \tag{5}$$

$$y''' + xy'' + 5y^2 = 2e^x \tag{6}$$

$$(y')^2 + 3xy = 1 \tag{7}$$

$$a^2 u_{xx} = u_{tt} - k u_t \tag{8}$$

$$u_{xy} = 0 \tag{9}$$

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0 \tag{10}$$

Remark. Various notations for derivatives are commonly employed, depending upon which is convenient at the time. For instance, we recognize the equivalences $y' = dy/dx$, $y'' = d^2y/dx^2, \ldots, y^{(n)} = d^n y/dx^n$. For partial derivatives, comparable notation is $u_x = \partial u/\partial x$, $u_{xy} = \partial^2 u/\partial y\, \partial x$, and so forth.

In order to provide a framework in which to discuss various solution techniques for DEs, it is helpful to first introduce **classification schemes** for the equations. For example, if the unknown function y appearing in a DE depends on only a single independent variable, say x, the equation is said to be an **ordinary differential equation** (ODE). Most of the DEs in this text are ODEs. When the unknown function depends upon more than one independent variable, the derivatives will be partial derivatives and the equation is then called a **partial differential equation** (PDE). Examples of ODEs are given by (3) through (7) above, while (8) through (10) are PDEs.