Introduction to DIFFERENTIAL EQUATIONS with Boundary Value Problems

Larry C. Andrews

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Sponsoring Editor: Peter Coveney/George Duda Project Editor: Ellen MacElree/Kristin Syverson Art Direction: Heather A. Ziegler/Julie Anderson

Text Adaptation: Heather A. Ziegler Cover Coordinator: Julie Anderson Cover Design: Matthew J. Doherty Cover Photo: © Pete Turner Text Art: Syntax International

Photo Research: Nina Page (cover)
Director of Production: Jeanie A. Berke
Production Assistant: Linda Murray
Compositor: Syntax International

Printer/Binder: R. R. Donnelley & Sons Company

Cover Printer: Lehigh Press Lithographers

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Library of Congress Cataloging-in-Publication Data

Andrews, Larry C.

Introduction to differential equations with boundary value problems

Larry C. Andrews

p. cm.

Includes index.

ISBN 0-06-040293-8

1. Differential equations. 2. Boundary value problems.

I. Title.

QA371.A58 1990

515'.35-dc20

90-37829 CIP

Introduction to DIFFERENTIAL EQUATIONS with Boundary Value Problems

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Preface

This book is intended as a one-semester or two-semester introductory treatment of differential equations and their applications. It is designed for students in mathematics, engineering, or science who have successfully completed the basic sequence of courses in calculus.

Because of the importance of differential equations in a variety of engineering and science areas, there are a number of applications to problems in the physical sciences, as well as some in the social and life sciences, that are prominent throughout the text. However, I have tried to avoid the temptation of introducing a multitude of applications in many diverse areas of application. My experience is that students are often distracted by too many different types of applications. A few basic applications readily make the point about the important role of differential equations in the real world. Moreover, I have kept the discussions involving applications at an elementary level so that a minimal background in the various sciences is required of the student or instructor to follow or understand them. In addition, I have tried to maintain a close relationship between mathematical theories and applications, whenever possible, by providing physical interpretations of some of the mathematical results.

The text contains the standard material that is found in the majority of introductory texts on differential equations, but contains some distinctive features that we list below.

Distinctive Features

- Linear first-order equations: A separate discussion of the notions of homogeneous and particular solutions in addition to the standard integrating factor technique for solving first-order linear equations (Chapter 2). I believe this provides a unifying point of view that is useful in later chapters dealing with linear equations of higher order.
- Green's function: Another novel feature of the text is the introduction of the causal Green's function (Chapter 5) for handling nonhomogeneous initial value problems in a systematic and physically meaningful fashion. This function is linked in Chapter 6 to the impulse response function which is basic in engineering applications involving the analysis of linear systems.
- Qualitative methods: Also included is a brief discussion of the qualitative methods used in oscillation theory (Chapter 5) and the stability of solutions of nonlinear systems of equations (Chapter 9). These brief exposures to qualitative methods permit the student to see how the general behavior of solutions to certain differential equations can still be determined in the absence of an explicit solution function.

- Worked examples: There are over 220 numbered worked examples, each of which is generally indicative of typical problems to be found in the exercise sets.
- Exercises: Nearly 1800 problems are included in the exercise sets, containing a blend of drill-like problems, some more difficult and some that extend the theory and applications beyond that discussed in the exposition. Problems that are considered more challenging are marked by a star (*).

Each chapter in the text begins with an overview of the topics to be covered in that chapter. In addition, each chapter contains a chapter summary in which the most important chapter topics are highlighted. The first eight chapters, which make up the bulk of material covered in most one-semester courses, also contain a review exercise set at the end of the chapter summary. Many of the chapters after the first four chapters are independent of each other so that various arrangements of topics can be made to suit individual course needs. Also, sections that can easily be omitted for a shorter course are marked [0].

Answers to all odd-numbered problems are provided at the end of the text to aid the students, while an instructor's Answer Key is available which contains answers to all problems (both odd and even-numbered problems). In addition, there is an accompanying Solutions Manual which contains detailed solutions to every other odd problem plus those even-numbered problems marked by a rectangular black bullet ... Not included in the Solutions Manual are the review exercises at the end of the first eight chapters.

I am grateful to:

Harvey Greenwald California Polytechnic State University

San Luis Obispo

Gilbert Lewis Michigan Technological University

Joaquin Loustaunau New Mexico State University

Bernard Marshall

Allan Krall Pennsylvania State University

Seymour Goldberg University of Maryland

Maurino Bautista Rochester Institute of Technology

Hendrik Kuiper Arizona State University

who served as reviewers. Also, I wish to acknowledge the hard work of Jody DeVoe who worked out the answers to all the exercises. Finally, I wish to thank my editor Peter Coveney and the entire production staff of HarperCollins for the fine job they did in getting this text published in a timely manner.

L. C. Andrews

Contents

Preface xiii

CHAPTER 1 Basic Concepts	1
1.1 Introduction 2 1.2 Classification of DEs 3 1.2.1 Origin and Applications of DEs [0]† 1.2.2 Linear Approximation 6 1.3 Solutions of DEs 7	5
 1.3.1 Particular and General Solutions 1.4 Initial and Boundary Conditions 1.5 Chapter Summary 16 1.6 Historical Comments 17 	11 13
CHAPTER 2 First-Order Equa	tions 19

	Introduction 20 Separation of Variables 22
	2.2.1 Homogeneous Equations 25
2.3	Exact Equations 29
	2.3.1 Integrating Factors 34
2.4	Linear Equations—Part I 37
	2.4.1 Method of Integrating Factors 38
2.5	Linear Equations—Part II 42
[0]	2.5.1 Homogeneous Equations 43 2.5.2 Nonhomogeneous Equations 45 2.5.3 Physical Interpretations 50

 $^{^{\}dagger}\left[0\right]$ indicates an optional section.

[0]	2.6	Miscellaneous Topics 54
		2.6.1 Bernoulli's Equation 54
		2.6.2 Picard's Method 56
		Chapter Summary 59
	2.8	Historical Comments 61
	CL	AAPTER 3 Applications Involving First-Order Equations 63
	CI	Applications involving First-Order Equations 00
	2 1	Introduction 64
	_	Orthogonal Trajectories 65
		Problems in Mechanics and Simple Electric Circuits 68
		3.3.1 Free-Falling Bodies 68
		3.3.2 Velocity of Escape 72
		3.3.3 Simple Electric Circuits 74
	3.4	Growth and Decay 77
		3.4.1 Radioactive Decay and Half-Life 77
		3.4.2 Population Growth 79
	3.5	Miscellaneous Applications 84
		3.5.1 Cooling and Mixing Problems 84
		3.5.2 Flow of Water through an Orifice 86 3.5.3 Curves of Linear Pursuit 88
	3.6	Chapter Summary 93
	3.0	Chapter Summary 55
	CF	HAPTER 4 Linear Equations of Higher Order 95
	4.1	Introduction 96
		Second-Order Homogeneous Equations 98
		4.2.1 Fundamental Solution Sets 99
		4.2.2 Wronskians 103
	4.3	Constructing a Second Solution from a Known Solution 108
	4.4	Homogeneous Equations with Constant Coefficients 111
		4.4.1 Complex Exponential Functions 114
	_	4.4.2 Hyperbolic Functions 116
	4.5	Differential Operators 119
		4.5.1 Polynomial Operators 120
	[0]	4.5.2 Variable-Coefficient Operators 121

	4.6	Higher-Order Homogeneous Equations 723
		4.6.1 Fundamental Solution Sets 124
		4.6.2 Constant-Coefficient DEs 126
	4.7	Nonhomogeneous Equations—Part I 130
		4.7.1 Method of Undetermined Coefficients 131
	4.8	Nonhomogeneous Equations—Part II 140
		4.8.1 Variation of Parameters 140
	4.9	Cauchy-Euler Equations 147
		Chapter Summary 151
	4.11	Historical Comments 152
	CH	AAPTER 5 Initial Value Problems 154
	07	707
	51	Introduction 155
	-	Small Free Motions of a Spring-Mass System 156
		5.2.1 Undamped Motions 158
	[0]	5.2.2 The Pendulum Problem 162
	L - 3	5.2.3 Damped Motions 163
	5.3	Forced Motions of a Spring-Mass System 168
		5.3.1 Undamped Motions 168
		5.3.2 Damped Motions 171
		Simple Electric Circuits 176
[0]	5.5	Method of Green's Function 180
		5.5.1 One-Sided Green's Function 182
		5.5.2 Table of One-Sided Green's Functions 186
[0]	5.6	Elementary Oscillation Theory 188
		5.6.1 Zeros of Bessel Functions 193
	5.7	Chapter Summary 195
	CH	HAPTER 6 The Laplace Transform 198
	01	7.0 THE Euplace Hallstotti 150
	6 1	Introduction 199
		Laplace Transforms of Some Elementary Functions 200
	U.E.	6.2.1 Existence Theorem 202
	62	Operational Properties 206
	0.5	Operational Floderties 200

6.3.1 Shift Property

viii	CONTENTS
VIII	CONTENTS

		6.3.2 Transforms of Derivatives 209 6.3.3 Derivatives of Transforms 210
	6.4	Inverse Transforms 213 6.4.1 Operational Properties 214 6.4.2 Partial Fractions 216
	6.5	Initial Value Problems 220 6.5.1 Applications 223
	6.6 6.7	Convolution Theorem 228 Laplace Transforms of other Functions 232
		6.7.1 Periodic Functions 232 6.7.2 Discontinuous Functions 234
[0]	6.8	Impulse Functions 243
		6.8.1 Impulse Response Function 246 6.8.2 One-Sided Green's Function 247
	6.9 6.10	Chapter Summary 249 Historical Comments 250
	CH	APTER 7 Power Series Methods 251
	<u>CH</u>	APTER 7 Power Series Methods 251
	7.1	Introduction 252
	7.1 7.2	Introduction 252 Review of Power Series 252
	7.1 7.2	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255
	7.1 7.2 7.3	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256
	7.1 7.2 7.3	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259
	7.1 7.2 7.3	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260
	7.1 7.2 7.3	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260 Solutions near a Singular Point—Part I 271
	7.1 7.2 7.3	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260
	7.1 7.2 7.3	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260 Solutions near a Singular Point—Part I 271 7.4.1 Regular Singular Points: Method of Frobenius 273 7.4.2 Case I: Roots Differing by a Noninteger 276 Solutions Near a Singular Point—Part II 280
	7.1 7.2 7.3	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260 Solutions near a Singular Point—Part I 271 7.4.1 Regular Singular Points: Method of Frobenius 273 7.4.2 Case I: Roots Differing by a Noninteger 276
[0]	7.1 7.2 7.3 7.4	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260 Solutions near a Singular Point—Part I 271 7.4.1 Regular Singular Points: Method of Frobenius 273 7.4.2 Case I: Roots Differing by a Noninteger 276 Solutions Near a Singular Point—Part II 280 7.5.1 Case II: Equal Roots 280
[0]	7.1 7.2 7.3 7.4 7.5	Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260 Solutions near a Singular Point—Part I 271 7.4.1 Regular Singular Points: Method of Frobenius 273 7.4.2 Case I: Roots Differing by a Noninteger 276 Solutions Near a Singular Point—Part II 280 7.5.1 Case II: Equal Roots 280 7.5.2 Case III: Roots Differing by a Nonzero Integer 285
	7.1 7.2 7.3 7.4 7.5 7.6 7.7	Introduction 252 Review of Power Series 252 7.2.1 Shift of Index of Summation 255 Solutions near an Ordinary Point 256 7.3.1 Ordinary and Singular Points: Polynomial Coefficients 259 7.3.2 Ordinary and Singular Points: General Coefficients 259 7.3.3 General Method 260 Solutions near a Singular Point—Part I 271 7.4.1 Regular Singular Points: Method of Frobenius 273 7.4.2 Case I: Roots Differing by a Noninteger 276 Solutions Near a Singular Point—Part II 280 7.5.1 Case II: Equal Roots 280 7.5.2 Case III: Roots Differing by a Nonzero Integer 285 Legendre's Equation 292

7.7.3 Bessel Functions of the Second Kind 301

7.8	Chapter Summary	305
7.9	Historical Comments	306

CHAP	TFR	8	Linear	Systems o	f Equations	308
UIIMI	1 - 1			Ovatellia o	I Eddarions	

0 4	I make a discarding to	309
8 1	Introduction	.3(7.7

8.2 Theory of First-Order Systems 310

- 8.2.1 Fundamental Solution Sets 312
- 8.2.2 Nonhomogeneous Systems 314

8.3 The Operator Method 315

- 8.3.1 Determinant Formulation 317
- 8.3.2 Homogeneous Equations of First Order 319

8.4 Applications 322

- 8.4.1 Coupled Spring-Mass Systems 322
- 8.4.2 Electrical Networks 324
- 8.4.3 Mixing Problems 326
- 8.5 The Method of Laplace Transforms 329

[0] 8.6 Review of Matrices 332

- 8.6.1 Algebraic Properties 333
- 8.6.2 Matrix Functions 338

8.7 First-Order Systems—Matrix Formulation 340

- 8.7.1 Homogeneous Systems 342
- 8.7.2 Fundamental Matrix 345
- 8.7.3 Nonhomogeneous Systems 347

8.8 Homogeneous Systems with Constant Coefficients 349

- 8.8.1 Systems of Two Equations—Eigenvalue Method 351
- 8.8.2 Higher-Order Systems 359
- 8.9 Nonhomogeneous Systems 362
- [0] 8.10 The Matrix Exponential Function 365
 - 8.11 Chapter Summary 373

CHAPTER 9 Nonlinear Systems and Stability 376

- 9.1 Introduction 377
- 9.2 Systems of One Equation 377
- 9.3 Systems of Two Equations and the Phase Plane 380
 - 9.3.1 Critical Points and Stability 384

9.4 Stability of Linear Systems 3899.5 Stability of Nonlinear Systems 3

	9.5.1 Nonlinear Pendulum Problem 398 9.5.2 Predator-Prey Problem 401
9.6	Chapter Summary 404
CH	IAPTER 10 Numerical Methods 405
	Introduction 406 2 Simple Methods for First-Order Equations 406
	10.2.1 The Euler Method 408
	10.2.2 The Improved Euler Method 413
10.	10.2.3 Errors 417
	The Runge-Kutta Method 419 Systems of Equations 424
	10.4.1 Higher-Order Equations 425
10.5	Chapter Summary 427
	× ·
CH	VAPTER 11 Boundary Value Problems and Fourier Series 429
11.1	Introduction 430
11.2	P. Boundary Value Problems 430
	11.2.1 Deflections of an Elastic String 430
44 .	11.2.2 General Theory 433
11.3	B Eigenvalue Problems 439 11.3.1 Buckling of a Long Column 445
11 4	Fourier Series—Part 1 449
	11.4.1 Sine Series 449
	11.4.2 Cosine Series 452
11.5	Fourier Series—Part II 454
	11.5.1 Convergence of the Series 457
	11.5.2 Even and Odd Functions: Cosine and Sine Series 461 11.5.3 Periodic Extensions 464
11 6	
11.0	Sturm-Liouville Problems and Eigenfunction Expansions 470 11.6.1 Generalized Fourier Series 473
11 7	Chapter Summary 476
	B Historical Comments 477

CHAPTER 12 Applications Involving Partial Differential Equations 479

	Introduction 480 The Heat Equation 481
[0]	12.2.1 Homogeneous Boundary Conditions 482 12.2.2 Nonhomogeneous Boundary Conditions 486 12.2.3 Derivation of the Governing Equation 490
12.3	The Wave Equation 492
	12.3.1 Free Motions of a Vibrating String 492 12.3.2 d'Alembert's Solution 496
12.4	Laplace's Equation 500
	12.4.1 Rectangular Domains 501 12.4.2 Circular Domains 505
12.5	Generalized Fourier Series 509
	12.5.1 Convective Heat Transfer at One Endpoint 509 12.5.2 Nonhomogeneous Heat Equation 511

12.5.3 Applications Involving Bessel Functions

513

12.6 Chapter Summary 519 **12.7 Historical Comments** 520

[0]

References for Additional Reading 522
Answers to Odd-Numbered Problems 523
Index 541

CHAPTER 1

Basic Concepts

- 1.1 INTRODUCTION
- 1.2 CLASSIFICATION OF DEs
- 1.3 SOLUTIONS OF DEs

- 1.4 INITIAL AND BOUNDARY CONDITIONS
- 1.5 CHAPTER SUMMARY
- 1.6 HISTORICAL COMMENTS

Differential equations play a fundamental role in engineering, mathematical and physical sciences, and the life sciences because they can be used in the formulation of many physical laws and relations. The development of the theory of differential equations is closely interlaced with the development of mathematics in general, and it is indeed difficult to separate the two. In fact, most of the famous mathematicians from the time of Newton and Leibniz had some part in the cultivation of this fascinating subject.

In a systematic development of the general theory of differential equations it is helpful to organize the various types of equations into classes. The reason is that all differential equations belonging to a particular class can often be solved by the same method. Therefore, in Section 1.2 we start with a discussion of some of the various classification schemes, emphasizing order and linearity.

In Section 1.3 we define what we mean by a solution of a differential equation. Here we discover that differential equations are peculiar in that they generally possess many different solutions, and for this reason we usually seek a specific function, called a general solution, with the property that all solutions can be obtained from it.

We introduce the notions of initial value problem and boundary value problem in Section 1.4. These are the names attached to those problems occurring in applications wherein the solution of a differential equation must satisfy certain additional auxiliary conditions. The study of these problems is so important in applications that we have devoted a significant portion of the text to developing the theory associated with them. In the last section of the chapter are some historical comments concerning differential equations.

1.1 Introduction

The theory of differential equations (DEs) has played an important role in science and engineering since the invention of the calculus by Newton[†] and Leibniz.[‡] Problems in the physical sciences have subsequently been investigated primarily by formulating them as DEs. The first problems studied from this point of view came mostly from the field of mechanics.

The role of DEs in solving problems in mechanics is nicely illustrated by considering Newton's second law of motion. In beginning physics courses this law is commonly introduced by the simple algebraic formula

$$F = ma$$

For a single "particle" in motion, F denotes the force acting on the particle (which may be simply its weight), m is the mass of the particle (generally assumed constant), and a is its acceleration. In solving problems in mechanics, however, we are usually concerned also with the velocity and position of the particle as a function of time, not just with its acceleration. We may recall from calculus that acceleration is the time derivative of velocity v = v(t), that is, a = dv/dt. Hence, Newton's second law can also be expressed by the equation

$$F = m \frac{dv}{dt}$$

Moreover, if y = y(t) is the position of the particle at time t, then its velocity is related by v = dy/dt. Using this relation, we see that $dv/dt = d^2y/dt^2$, and Newton's second law now assumes the additional form

$$F = m \frac{d^2 y}{dt^2}$$

Because they involve derivatives of unknown quantities, these last two equations are examples of DEs. Clearly, DEs can evolve quite naturally in the formulation of even rather simple problems in physics. Of course, DEs are now used extensively in all areas of physics and engineering, and more recently have also found their way into the social and life sciences.

[†] Sir Isaac Newton (1642–1727) was born on Christmas Day in the countryside of England. Along with Leibniz, he is credited with the invention of the calculus (see also Section 1.6).

[‡] Gottfried Wilhelm von Leibniz (1646–1716) was born in Leipzig. Known as both a philosopher and mathematician, he is credited with building a remarkable system of modern philosophy and being codeveloper of the calculus (see also Section 1.6).

1.2 Classification of DEs

By a differential equation we mean simply an equation that is composed of a single unknown function and a finite number of its derivatives. One of the simplest examples that occurs early in the calculus is to find all functions y for which

$$y' = f(x) \tag{1}$$

where f(x) is a given function. For instance, if $f(x) = x^2$, the unknown function y is obtained through a simple integration to yield

$$y = \frac{x^3}{3} + C \tag{2}$$

where C is a constant of the integration which can assume any value.

Most of the DEs that concern us in this text are not of the simple variety as described by Equation (1). Typical examples, some of which we discuss in later chapters, include the following:

$$y' = x^2 y^3 \tag{3}$$

$$y'' + k^2 y = \sin x \tag{4}$$

$$y'' + b\sin y = 0 \tag{5}$$

$$y''' + xy'' + 5y^2 = 2e^x (6)$$

$$(y')^2 + 3xy = 1 (7)$$

$$a^2 u_{xx} = u_{tt} - k u_t \tag{8}$$

$$u_{xy} = 0 (9)$$

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0 ag{10}$$

Remark. Various notations for derivatives are commonly employed, depending upon which is convenient at the time. For instance, we recognize the equivalences y' = dy/dx, $y'' = d^2y/dx^2$, ..., $y^{(n)} = d^ny/dx^n$. For partial derivatives, comparable notation is $u_x = \partial u/\partial x$, $u_{xy} = \partial^2 u/\partial y \partial x$, and so forth.

In order to provide a framework in which to discuss various solution techniques for DEs, it is helpful to first introduce **classification schemes** for the equations. For example, if the unknown function y appearing in a DE depends on only a single independent variable, say x, the equation is said to be an **ordinary differential equation** (ODE). Most of the DEs in this text are ODEs. When the unknown function depends upon more than one independent variable, the derivatives will be partial derivatives and the equation is then called a **partial differential equation** (PDE). Examples of ODEs are given by (3) through (7) above, while (8) through (10) are PDEs.