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Regularity Properties of Functional Equations in Several Variables

Antal Járai

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This book is dedicated to János Aczél, my "mathematical grandfather", the teacher of several of us in the field of functional equations, and to my teacher Zoltán Daróczy who introduced me to functional equations.

PREFACE

This book is about regularity properties of functional equations. It contains, in a unified fashion, most of the modern results about regularity of non-composite functional equations with several variables. It also contains several applications including very recent ones. I hope that this book makes these results more accessible and easier to use for everyone working with functional equations.

This book could not have been written without the stimulating atmosphere of the International Symposium on Functional Equations conference series and thus I am grateful to all colleagues working in this field. This series of conferences was created by János Aczél. I am especially grateful to him for inviting me to the University of Waterloo, Canada, which provided a peaceful working environment. I started this book in September 1998 during my stay in Waterloo.

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Between 1974 and 1997 I worked at Kossuth Lajos University, Debrecen, Hungary. Naturally, I am grateful to all members of the Debrecen School of functional equations.

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PRELIMINARIES

In this chapter, starting with simple examples, we describe the problems with which we will deal in this book. We also present simple examples of our methods. First we formulate the fundamental problem, then analyse its conditions and explore its applicability. We then formulate theorems that follow from our results as corollaries to that fundamental problem. Then we survey possibilities for generalization. We close this chapter by summarizing our notation and terminology, including the formulation of theorems not readily available in the literature or usually formulated in a different way.

1. INTRODUCTION

1.1. General considerations and simple examples. As a first, illustrative example let us consider the best-known functional equation, *Cauchy's equation*

$$f(x+y) = f(x) + f(y) \tag{1}$$

with unknown function f. In a wider sense differential equations, integral equations, variational problems, etc. are also functional equations, but here we will use this expression in a more restrictive sense for functional equations without infinitesimal operations such as integration and differentiation. For a more formal definition, see Aczél [3], 0.1. To formulate a functional equation exactly we have to give the set of functions in which we look for solutions. We also have to give the *domain* of the functional equation. In the above example this is the set of the pairs (x, y) of the *variables* x and y for which equality has to be satisfied. For example, we may look for all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that (1) is satisfied for all $(x, y) \in [0, \infty[\times \mathbb{R}$. Conditions such as measurability, Baire property, continuity everywhere or in a point, boundedness, differentiability, analyticity, etc. are called *regularity conditions*. If this kind of conditions are imposed on the solution, then we say that we look for *regular solutions*. Otherwise, if we look for solutions among all maps from a given set into another given set, then we say that we look for the *general solution* of the functional equation.

Usually, the *domain* of the functional equation is the set of all tuples of the variables for which both sides are defined. For example, if we say that $f : \mathbb{R} \to \mathbb{R}$ is a solution of Cauchy's functional equation, then it is implicitly understood that (1) is satisfied for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. If the domain of the equation is not the largest possible for which both sides are defined, then we speak about an equation with *restricted domain*; the term *conditional equation* is also used, especially if the domain of the equation also depends on the solution or solutions.

Cauchy's equation is a functional equation with two variables; the variables denoted by x and y in (1). Equations like $f(x) = f(-x)$, $f(x) = -f(-x)$, $f(2x) = f(x)^2$, or difference equations are called functional equations in a *single variable*. The "single variable" may also be a vector variable; it is understood that there are no more variables in the equation than the number of places in the unknown function or the minimal number of places in the unknown functions — if there is more than one. Otherwise we speak about a functional equation in *several variables*. This distinction is very useful in practice. There is a large difference between functional equations with a single variable and several variables: the methods used in the two cases are quite different. In this book we deal with functional equations in several variables. About equations in a single variable see the books Kuczma [126] and Kuczma, Choczewski, Ger [128].

The distinction between functional equations in a single variable and in several variables and what we have said about variables, domain, regular and general solutions also apply to *systems of functional equations*.

Further simple examples of functional equations are *Cauchy's exponen-*

tial equation

$$f(x+y) = f(x)f(y), \tag{2}$$

Cauchy's power equation

$$f(xy) = f(x)f(y), \tag{3}$$

and *Cauchy's logarithmic equation*

$$f(xy) = f(x) + f(y). \tag{4}$$

Observe, that solutions f of (2) mapping $]0,\infty[$ into the normed algebra of all bounded linear operators on a Banach space gives operator semigroups. The usefulness of semigroups in the study of evolution equations such as the heat equation or Schrödinger's equation is well known, see for example Hille and Phillips [72]. The overall importance of equations (1)–(4) is due to the fact that they describe *homomorphisms*.

We move toward a *general theory of functional equations*, and we do not intend to study specific functional equations, except as examples, even if they are very important.

It is a well-known phenomenon that one functional equation can determine several unknown functions. This is the situation, for example, for the analogue of Cauchy's functional equation with several unknown functions which is called *Pexider's equation*:

$$f_1(x+y) = f_2(x) + f_3(y). \tag{5}$$

Indeed, if $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$, by putting $y = 0$ and $x = 0$ in (5) we may express f_2 and f_3 by f_1, respectively. By putting $x = 0$ and $y = 0$ simultaneously in (5) and using the resulting relation we obtain that $f = f_1 - f_1(0)$ satisfies Cauchy's equation (1). Hence (5) can be reduced to (1). Similar phenomena occur often when different occurrences of the unknown function f are replaced by f_1, f_2, etc., a process sometimes called "*Pexiderization*".

Jensen's equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \tag{6}$$

can be considered as a special case of Pexider's equation, and we obtain that an $f : \mathbb{R} \to \mathbb{R}$ function is a solution of (6) if and only if the function $f - f(0)$ satisfies (1).

It is also possible to consider *functional inequalities*. Functional inequality

$$f(x+y) \leq f(x) + f(y) \tag{7}$$

related to Cauchy's equation describes *subadditive* functions and functions satisfying

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{8}$$

are the so-called *Jensen convex* functions.

We will use the above simple functional equations as illustrative examples. Their detailed study can be found in the book of Aczél [3] or in the book of Aczél and Dhombres [20].

1.2. Simple examples: smooth solutions. Let us suppose that a solution $f : \mathbb{R} \to \mathbb{R}$ of Cauchy's equation $f(x+y) = f(x) + f(y)$ is *analytic*. Substituting $y = x$ we obtain the equation $f(2x) = 2f(x)$ in a single variable $x \in \mathbb{R}$. Analyticity is such a strong regularity condition that even this single variable equation has not too many analytic solutions. For the solution $f(x) = c_0 + c_1 x + \cdots$ we obtain

$$c_0 + 2c_1 x + 4c_2 x^2 + \cdots = 2c_0 + 2c_1 x + 2c_2 x^2 + \cdots$$

in a neighborhood of the origin, and hence that the solution can only be $f(x) = cx$ with an arbitrary constant $c = c_1$. Substitution shows that this is indeed a solution of Cauchy's equation.

The case of Cauchy's exponential equation $f(x+y) = f(x)f(y)$ is much more interesting. As above, we obtain the single variable equation $f(2x) = f(x)^2$, and, if $f : \mathbb{R} \to \mathbb{R}$ is analytic, $f(x) = c_0 + c_1 x + \cdots$, then

$$c_0 + 2c_1 x + 4c_2 x^2 + \cdots = c_0^2 + 2c_0 c_1 x + (2c_0 c_2 + c_1^2)x^2 + \cdots .$$

Hence $c_0 = c_0^2$. There are two possibilities. The first is that $c_0 = 0$, which implies that $c_n = 0$ for each n, and hence $f \equiv 0$. The second is that $c_0 = 1$. In this case c_1 could be arbitrary, and from the equation

$$2^n c_n = \sum_{i=0}^{n} c_i c_{n-i},$$

using the notation $c = c_1$, we obtain by induction that $c_n = c^n/n!$. Hence all analytic solutions are given by $f(x) = \sum_{n=0}^{\infty} c^n x^n/n! = \exp(cx)$. The same method gives complex analytic solutions $f : \mathbb{C} \to \mathbb{C}$, too. Let us observe that this is a nice way to introduce exponential functions (and hence the related functions sin, cos, sinh, and cosh) using only the most important property of exponentiation. Note that this was Cauchy's original motivation to investigate functional equations 1.1.(1)–1.1.(4): he wanted to avoid "circulus vitiosus" by studying power functions; see the historical remarks in the book of Aczél and Dhombres [20], pp. 365–371.

Now let us only suppose that the solution $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable. In the case of Cauchy's equation, $f(x+y) = f(x) + f(y)$, let us differentiate both sides with respect to y. This "kills" the first term on the right-hand side, and we obtain that $f'(x+y) = f'(y)$ for every $x, y \in \mathbb{R}$. Differentiating again, but with respect to x we can "kill" the other term on the right-hand side, too, and we obtain $f''(x+y) = 0$. Substituting $y = 0$ we have $f''(x) = 0$, a differential equation. All solutions of this equation have the form $f(x) = c_0 + cx$, $c_0, c \in \mathbb{R}$. Substituting this into the original functional equation we see that $c_0 = 0$, and we obtain that twice differentiable solutions are exactly the functions $f(x) = cx$.

This simple example illustrates a general method to get "smooth enough" solutions. The general tactic is to "kill" some terms by applying appropriate differential operators, and to obtain differential equations by appropriate substitutions. Usually, appropriate substitutions or use of certain symmetries of the equation results in a differential equation with lower degree. For example, substituting $y = 0$ in the equation $f'(x+y) = f'(y)$ we obtain that $f'(x) = c$ with $c = f'(0)$, a first order equation. Cauchy's exponential equation, $f(x+y) = f(x)f(y)$, similarly yields $f'(x+y) = f(x)f'(y)$, and after substituting $y = 0$ we obtain $f'(x) = cf(x)$, where $c = f'(0)$.

Let us observe that in both cases, the general once differentiable solution $f : \mathbb{R} \to \mathbb{R}$ is the same as the general analytic solution.

1.3. Simple examples: regularity properties. How to obtain solutions of the above examples, Cauchy's equation and Cauchy's exponential equation under much weaker regularity assumptions? A general way is to prove that weak regularity conditions, say continuity or measurability of solutions implies much stronger regularity conditions, their differentiability or even analyticity. For example, let us observe that in both of the above cases the differential equation obtained for the solutions in the previous point implies directly that the solutions are analytic (see Dieudonné [49], 10.5.3).

If we have a continuous solution $f : \mathbb{R} \to \mathbb{R}$, then integrating Cauchy's equation over an interval $[a, b]$ of positive length yields

$$f(x)(b-a) = \int_a^b f(x+y)\,dy - \int_a^b f(y)\,dy.$$

Substituting a new variable $u = x + y$ we obtain

$$f(x) = \frac{1}{b-a}\int_{x+a}^{x+b} f(u)\,du - \frac{1}{b-a}\int_a^b f(y)\,dy.$$

The right-hand side is differentiable, so we obtain that f is differentiable. If we want to deduce that f is twice differentiable, we can apply the same reasoning to the equation $f'(x+y) = f'(x)$ obtained by differentiation with respect to x from the original. Higher order differentiability can be obtained analogously.

In the case of Cauchy's exponential equation $f \equiv 0$ is one of the continuous solutions $f : \mathbb{R} \to \mathbb{R}$. If $f(y_0) \neq 0$, then we can choose a neighborhood $[a, b]$ of y_0 such that $f(y)/f(y_0) \geq 1/2$ for each $y \in [a, b]$. Integrating we obtain

$$f(x)\int_a^b f(y)\,dy = \int_a^b f(x+y)\,dy,$$

and hence that

$$f(x) = \frac{\int_{a+x}^{b+x} f(u)\,du}{\int_a^b f(y)\,dy}.$$

This implies that f is differentiable. Here, again, applying the same method for the equation $f'(x+y) = f'(x)f(y)$ obtained from the original equation by differentiation with respect to x gives that the solutions are twice differentiable, etc.

Now, let us consider a measurable solution $f : \mathbb{R} \to \mathbb{R}$ of Cauchy's equation. Let $[a, b]$ be an interval with positive length η. Let $x_0 \in \mathbb{R}$ be arbitrary. By Lusin's theorem there exists a compact set C_1 contained in $[x_0 + a, x_0 + b]$ and having Lebesgue measure greater than $3\eta/4$ such that $f|C_1$ is continuous. If $|x - x_0| < \eta/8$, then the set $C_1 - x$ is contained in $C = [a - \eta/8, b + \eta/8]$. Since the Lebesgue measure of $C \setminus (C_1 - x)$ and $C \setminus (C_1 - x_0)$ are less than $\eta/2$, they cannot cover C. Hence the intersection $(C_1 - x) \cap (C_1 - x_0)$ is nonvoid. Now, let $\varepsilon > 0$ be arbitrary. Since $f|C_1$ is uniformly continuous, there exists a $\delta > 0$ such that if $u, u' \in C_1$ then $|f(u) - f(u')| < \varepsilon$. Hence, if $|x - x_0| < \min\{\eta/8, \delta\}$ then for any $y \in (C_1 - x) \cap (C_1 - x_0)$ we obtain

$$|f(x) - f(x_0)| \leq |f(x+y) - f(x_0+y)| + |f(y) - f(y)| < \varepsilon,$$

i. e., f is continuous at x_0. Since x_0 was arbitrary, f is continuous everywhere. The same method can be applied to Cauchy's exponential equation after introducing the new variable $t = x + y$ instead of x, i. e., to the equation $f(t) = f(t-y)f(y)$.

Note that Cauchy's logarithmic equation $f(xy) = f(x) + f(y)$ has no other solution $f : \mathbb{R} \to \mathbb{R}$ than $f \equiv 0$; this follows by substituting $y = 0$.

In the case of Cauchy's power equation $f(xy) = f(x)f(y)$ there are solutions $f : \mathbb{R} \to \mathbb{R}$ which are measurable but non-continuous, continuous but non-differentiable, etc. Indeed, the functions $x \mapsto |x|^c$ and $x \mapsto |x|^c \operatorname{sgn} x$ are solutions for any $c \in \mathbb{R}$ if 0^c is understood as 0.

1.4. Hilbert's fifth problem. In his celebrated address to the 1900 International Congress of Mathematicians, in his fifth problem Hilbert ([70] p. 304) asked[1]

> "*... how far Lie's concept of continuous groups of transformations is approachable in our investigations without the assumption of differentiability of the functions*"

More precisely[2]:

> "... hence there arises the question whether, through the introduction of suitable new variables and parameters, the group can always be transformed into one whose defining functions are differentiable ...".

Explaining that the group property is connected to a system of functional equations, in the second part of his fifth problem Hilbert goes on as follows[3]:

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[1] "*... inwieweit der Liesche Begriff der kontinuierlichen Transformationsgruppe auch ohne Annahme der Differenzierbarkeit der Funktionen unserer Untersuchung zugänglich ist.*"

[2] "... es ensteht mithin die Frage, ob nicht etwa durch Einführung geeigneter neuer Veränderlicher und Parameter die Gruppe stets in eine solche übergeführt werden kann, für welche die definierenden Funktionen differenzierbar sind, ...".

[3] "Überhaupt werden wir auf das weite und nicht uninteressante Feld der Funktionalgleichungen geführt, die bisher meist nur unter der Voraussetzung der Differenzierbarkeit der auftretenden Funktionen untersucht worden sind. Insbesondere die von ABEL (Werke, Bd. 1, S. 1, 61, 389) mit so vielem Scharfsinn behandelten Funktionalgleichungen, die Differenzengleichungen und andere in der Literatur vorkommende Gleichungen weisen an sich nichts auf, was zur Forderung der Differenzierbarkeit der auftretenden Funktionen zwingt, und bei gewissen Existenzbeweisen in der Variationsrechnung fiel mir direkt die Aufgabe zu, aus dem Bestehen einer Differenzengleichung die Differenzierbarkeit der betrachteten Funktionen beweisen zu müssen. In allen diesen Fällen erhebt sich daher die Frage, *inwieweit etwa die Aussagen, die wir im Falle der Annahme differenzierbarer Funktionen machen können, unter geeigneten Modifikationen ohne diese Voraussetzung gültig sind.*"

> "Moreover, we are thus led to the wide and interesting field of functional equations which have been heretofore investigated usually only under the assumption of the differentiability of the functions involved. In particular the functional equations treated by Abel (Oeuvres, vol. 1, pp. 1, 61, 389) with so much ingenuity, the difference equations, and other equations occurring in the literature of mathematics, do not directly involve anything which necessitates the requirements of the differentiability of the accompanying functions. In the search for certain existence proofs in the calculus of variations I came directly upon the problem: To prove the differentiability of the function under consideration from the existence of a difference equation. In all these cases, then, the problem arises: *In how far are the assertions which we can make in the case of differentiable functions true under proper modifications without this assumption?*"

(Hilbert's emphases.) After this Hilbert quotes a result of Minkowski which states that under certain conditions the solutions of the functional inequality

$$f(x+y) \leq f(x) + f(y) \qquad x, y \in \mathbb{R}^n$$

are partially differentiable, and remarks that certain functional equations, for example the system of functional equations

$$f(x+\alpha) - f(x) = g(x),$$
$$f(x+\beta) - f(x) = 0,$$

where α, β are given real numbers, may have solutions f which are continuous but non-differentiable, even if the given function g is analytic.

In our present-day language, it is customary to formulate the fifth problem of Hilbert as the question whether a locally Euclidean topological group is a Lie group. However, in the second part of his fifth problem, Hilbert draws attention to more general problems which today are called regularity problems. They require to prove that differentiability assumptions for functional equations, differential equations, and other equations can be replaced by much weaker assumptions (possibly with appropriate modifications of the problem). This idea returns in problems nineteen and twenty of Hilbert concerning calculus of variation and partial differential equations. See the book of Zeidler [209], II/A, pp. 86–93. As a general reference about Hilbert's problems, see the book [26] edited by Alexandrov.