

Value-at-Risk

THEORY

and

PRACTICE



Glyn A. Holton

VALUE-AT-RISK

Theory and Practice

GLYN A. HOLTON

Contingency Analysis, Boston



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*To the memory of
Elizabeth Anne Carlson*

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Part I

Overview

Preface

0.1. WHAT WE'RE ABOUT

A watershed in the history of value-at-risk (VaR) was the publication of JP Morgan's *RiskMetrics Technical Document*. Writing in the third edition of that document, Guldumann (1995) identified three practical methods for calculating VaR:

- the parametric method,
- the historical simulation method, and
- the structured Monte Carlo method;

and so, the “methods” approach for describing VaR was born. Explaining VaR as comprising three “methods” is simple, intuitive and direct. The “methods” approach has been widely adopted by authors of books and research papers. Variations on the three basic “methods” abound, but only one truly new “method” has been introduced since 1995. This might be termed the “quadratic method.” Rouvinez (1997) ultimately published it.

For some time, I have felt that the top-down “methods” approach for explaining VaR was flawed. It is like explaining options pricing theory by presenting the Black–Scholes (1973), Merton (1973), and Black (1976) option pricing formulas. As an alternative, a bottom-up description might start with arbitrage pricing, stochastic calculus and replicating portfolios. Top-down explanations are appealing because they go directly to results, but they lead nowhere after that. Bottom-up explanations build a foundation for deeper understanding and further research. I have felt that VaR required such a foundation.