Günter Hofstetter
Günther Meschke
Editors

International Centre for Mechanical Sciences

Numerical Modeling of Concrete Cracking

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NUMERICAL MODELING OF CONCRETE CRACKING

EDITED BY

GÜNTER HOFSTETTER
UNIVERSITY OF INNSBRUCK, AUSTRIA

GÜNTHER MESCHKE
RUHR UNIVERSITY BOCHUM, GERMANY

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PREFACE

Reliable model-based prognoses of the initiation and propagation of cracks in concrete plays an important role for the durability and integrity assessment of concrete and reinforced concrete structures. To this end, a large number of material models for concrete cracking based on different theories (e.g., damage mechanics, fracture mechanics, plasticity theory and combinations of the mentioned theories) as well as advanced finite element methods suitable for the representation of cracks (e.g., the Extended Finite Element Method and Embedded Crack Models) have been developed in recent years.

The focus of the Advanced School on "Numerical Modeling of Concrete Cracking" at the International Centre for Mechanical Sciences (CISM) at Udine in May 2009 was laid on numerical models for describing crack propagation in concrete and their applications to numerical simulations of concrete and reinforced concrete structures. The lectures of this course formed the basis for this book. Its aim is to impart fundamental knowledge of the underlying theories of the different approaches for modelling cracking of concrete and to provide a critical survey of the state-of-the-art in computational concrete mechanics.

This book covers a relatively broad spectrum of topics related to modelling of cracks, including continuum-based and discrete crack models, meso-scale models, advanced discretization strategies to capture evolving cracks based on the concept of finite elements with embedded discontinuities and on the extended finite element method, respectively, and, last but not least, extensions to coupled problems such as hygro-mechanical problems as required in computational durability analyses of concrete structures.

Innsbruck and Bochum,
March 2011,

Günter Hofstetter
University of Innsbruck
Austria

Günther Meschke
Ruhr-University Bochum
Germany

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Damage and Smeared Crack Models

Milan Jirásek
Czech Technical University in Prague, Czech Republic

1 Isotropic Damage Models

Continuum damage mechanics is a constitutive theory that describes the progressive loss of material integrity due to the propagation and coalescence of microcracks, microvoids, and similar defects. These changes in the microstructure lead to a degradation of material stiffness observed on the macroscale. The term "continuum damage mechanics" was first used by Hult in 1972 but the concept of damage had been introduced by Kachanov already in 1958 in the context of creep rupture (Kachanov, 1958) and further developed by Rabotnov (1968); Hayhurst (1972); Leckie and Hayhurst (1974). The simplest version of the isotropic damage model considers the damaged stiffness tensor as a scalar multiple of the initial elastic stiffness tensor, i.e., damage is characterized by a single scalar variable. A general isotropic damage model should deal with two scalar variables corresponding to two independent elastic constants of standard isotropic elasticity. More refined theories take into account the anisotropic character of damage; they represent damage by a family of vectors (Krajcinovic and Fonseka, 1981), by a second-order tensor (Vakulenko and Kachanov, 1971) or, in the most general case, by a fourth-order tensor (Chaboche, 1979). Anisotropic formulations can be based on the principle of strain equivalence (Lemaitre, 1971), or on the principle of energy equivalence (Cordebois and Sidoroff, 1979) (the principle of stress equivalence is also conceptually possible but is rarely used).

In the present chapter, we will focus on isotropic damage models and on smeared crack models, which incorporate anisotropy in a simplified way. Anisotropic damage models based on tensorial description of damage will are treated e.g. in Lemaitre and Desmorat (2005).

1.1 One-Dimensional Damage Model

Damage models work with certain internal variables that characterize the density and orientation of microdefects. To introduce the basic concepts, we start from the case of uniaxial stress. For the present purpose, the material is idealized as a

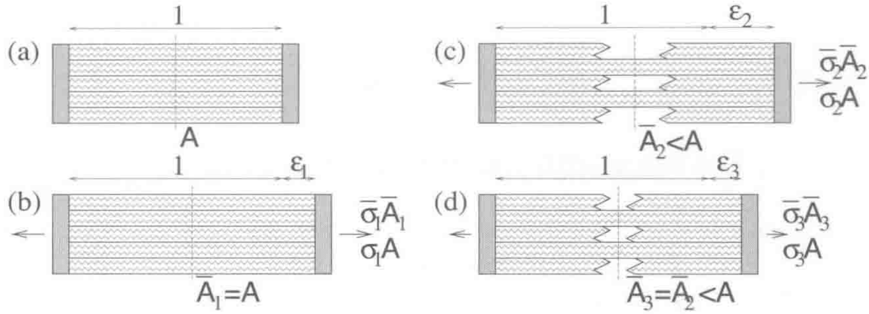

Figure 1. Representation of a uniaxial damage model as a bundle of parallel elastic fibers breaking at different strain levels

bundle of fibers parallel to the direction of loading (Fig. 1a). Initially, all the fibers respond elastically, and the stress is carried by the total cross section of all fibers, A (Fig. 1b). As the applied strain is increased, some fibers start breaking (Fig. 1c). Each fiber is assumed to be perfectly brittle, which means that the stress in the fiber drops down to zero immediately after a critical strain level is reached. However, since the critical strain is different for each fiber, the effective area $\bar{A}$ (i.e., the area of unbroken fibers that can still carry stress) decreases gradually from $\bar{A} = A$ to $\bar{A} = 0$. We have to make a distinction between the *nominal stress* σ, defined as the force per unit initial area of the cross section, and the *effective stress* $\bar{\sigma}$, defined as the force per unit effective area. The nominal stress enters the Cauchy equations of equilibrium on the macroscopic level, while the effective stress is the "true" stress acting in the material microstructure.[1] From the condition of equivalence, $\sigma A = \bar{\sigma}\bar{A}$, we obtain

$$\sigma = \frac{\bar{A}}{A}\bar{\sigma} \tag{1}$$

The ratio of the effective area to the total area, $\bar{A}/A$, is a scalar characterizing the *integrity* of the material. In damage mechanics it is customary to work with the *damage variable* defined as

$$D = 1 - \frac{\bar{A}}{A} = \frac{A - \bar{A}}{A} = \frac{A_d}{A} \tag{2}$$

where $A_d = A - \bar{A}$ is the damaged part of the area. An intact (undamaged) material is characterized by $\bar{A} = A$, i.e., by $D = 0$. Due to propagation of microdefects and

[1] Of course, detailed micromechanical analysis would reveal local oscillations of the stress fields dependent on the specific defect geometry, and the representation of the actual stress distribution by one averaged value—the effective stress—is just a simplification for modeling purposes.

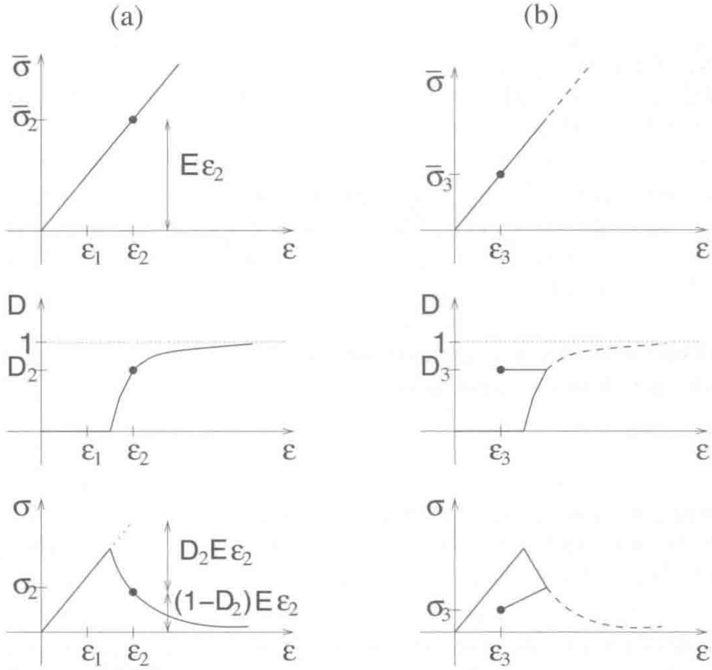

Figure 2. Evolution of effective stress $\bar{\sigma}$, damage variable D and nominal stress σ under a) monotonic loading, b) non-monotonic loading

their coalescence, the damage variable grows and at late stages of the degradation process it attains or asymptotically approaches the limit value $D = 1$, corresponding to a completely damaged material with effective area reduced to zero. In the simplest version of the model, each fiber is supposed to remain linear elastic up to the strain level at which it breaks.[2] Consequently, the effective stress $\bar{\sigma}$ is governed by Hooke's law,

$$\bar{\sigma} = E\varepsilon \tag{3}$$

Combining (1)–(3) we obtain the constitutive law for the nominal stress,

$$\sigma = (1 - D)E\varepsilon \tag{4}$$

Damage evolution can be characterized by the dependence of the damage variable on the applied strain,

$$D = g(\varepsilon) \tag{5}$$

[2] In general, the fictitious "fibers" can obey any (nonlinear) constitutive law, which provides one possible framework for coupling of damage with other dissipative phenomena, such as plasticity.

Function g affects the shape of the stress-strain diagram and can be directly identified from a uniaxial test. The evolution of the effective stress, damage variable, and nominal stress in a material that remains elastic up to the peak stress is shown in Fig. 2a. This description is valid only for monotonic loading by an increasing applied strain ε. When the material is first stretched up to a certain strain level ε_2 that induces damage $D_2 = g(\varepsilon_2)$ and then the strain decreases (Fig. 1d), the damaged area remains constant and the material responds as an elastic material with a reduced Young's modulus $E_2 = (1 - D_2)E$. This means that, during unloading and partial reloading, the damage variable in (4) must be evaluated from the largest previously reached strain and not from the current strain ε. It is convenient to introduce an internal variable κ characterizing the maximum strain level reached in the previous history of the material up to a given time t, i.e., to set

$$\kappa(t) = \max_{\tau \leq t} \varepsilon(\tau) \tag{6}$$

where t is not necessarily the physical time—it can be any monotonically increasing parameter controling the loading process. The damage evolution law (5) is then replaced by equation

$$D = g(\kappa) \tag{7}$$

that remains valid not only during monotonic loading but also during unloading and reloading. The evolution of the effective stress, damage variable, and nominal stress in a non-monotonic test is shown in Fig. 2b. Note that, upon a complete removal of the applied stress, the strain returns to zero (due to elasticity of the yet unbroken fibers), i.e., the pure damage model does not take into account any permanent strains. Nevertheless, the material state is different from the initial virgin state, because the damage variable is not zero and the stiffness and strength mobilized in a new tensile loading process are smaller than their initial values. The loading history is reflected by the value of the damage variable D.

To gain further insight, we rewrite the constitutive law (4) in the form $\sigma = E_s\varepsilon$ where $E_s = (1 - D)E$ is the apparent (damaged) modulus of elasticity. Instead of defining the variable κ through (6), we introduce a loading function $f(\varepsilon, \kappa) = \varepsilon - \kappa$ and postulate the loading-unloading conditions in the Kuhn-Tucker form,

$$f \leq 0, \qquad \dot{\kappa} \geq 0, \qquad \dot{\kappa} f = 0 \tag{8}$$

The first condition means that κ can never be smaller than ε, and the second condition means that κ cannot decrease. Finally, according to the third condition, κ can grow only if the current values of ε and κ are equal.

The basic ingredients of the uniaxial damage theory are summarized as follows:

- the stress-strain law in the secant format,

$$\sigma = E_s\varepsilon \tag{9}$$

- the equation relating the apparent stiffness to the damage variable,

$$E_s = (1 - D)E \tag{10}$$

- the law governing the evolution of the damage variable,

$$D = g(\kappa) \tag{11}$$

- the loading function

$$f(\varepsilon, \kappa) = \varepsilon - \kappa \tag{12}$$

specifying the elastic domain $\mathcal{E}_\kappa = \{\varepsilon \,|\, f(\varepsilon, \kappa) < 0\}$, i.e., the set of states for which damage does not grow, and
- the loading-unloading conditions (8).

1.2 Damage Models with Strain-Based Loading Functions

Simple Models with One Damage Variable. The simplest extension of the uniaxial damage theory to general multiaxial stress states is achieved by the isotropic damage model with a single scalar variable. Isotropic damage models are based on the simplifying assumption that the stiffness degradation is isotropic, i.e., stiffness moduli corresponding to different directions decrease proportionally, independently of the direction of loading. Since an isotropic elastic material is characterized by two independent elastic constants, a general isotropic damage model should deal with two damage variables. The model with a single variable makes use of an additional assumption that the relative reduction of all the stiffness coefficients is the same, in other words, that the Poisson ratio is not affected by damage. Consequently, the damaged stiffness tensor is expressed as

$$\mathbb{E}_S = (1 - D)\,\mathbb{E} \tag{13}$$

where $\mathbb{E}$ is the elastic stiffness tensor of the intact material, and D is the damage variable. In the present context, $\mathbb{E}_S$ is the secant stiffness that relates the total strain to total stress, according to the formula

$$\boldsymbol{\sigma} = \mathbb{E}_S : \boldsymbol{\varepsilon} = (1 - D)\,\mathbb{E} : \boldsymbol{\varepsilon} \tag{14}$$

Clearly, (13) is a generalization of (10), and (14) is a generalization of (9) and (4). In terms of the *effective stress tensor*, defined as

$$\overline{\boldsymbol{\sigma}} = \mathbb{E} : \boldsymbol{\varepsilon} \tag{15}$$

equation (14) can alternatively be written as

$$\boldsymbol{\sigma} = (1 - D)\,\overline{\boldsymbol{\sigma}} \tag{16}$$

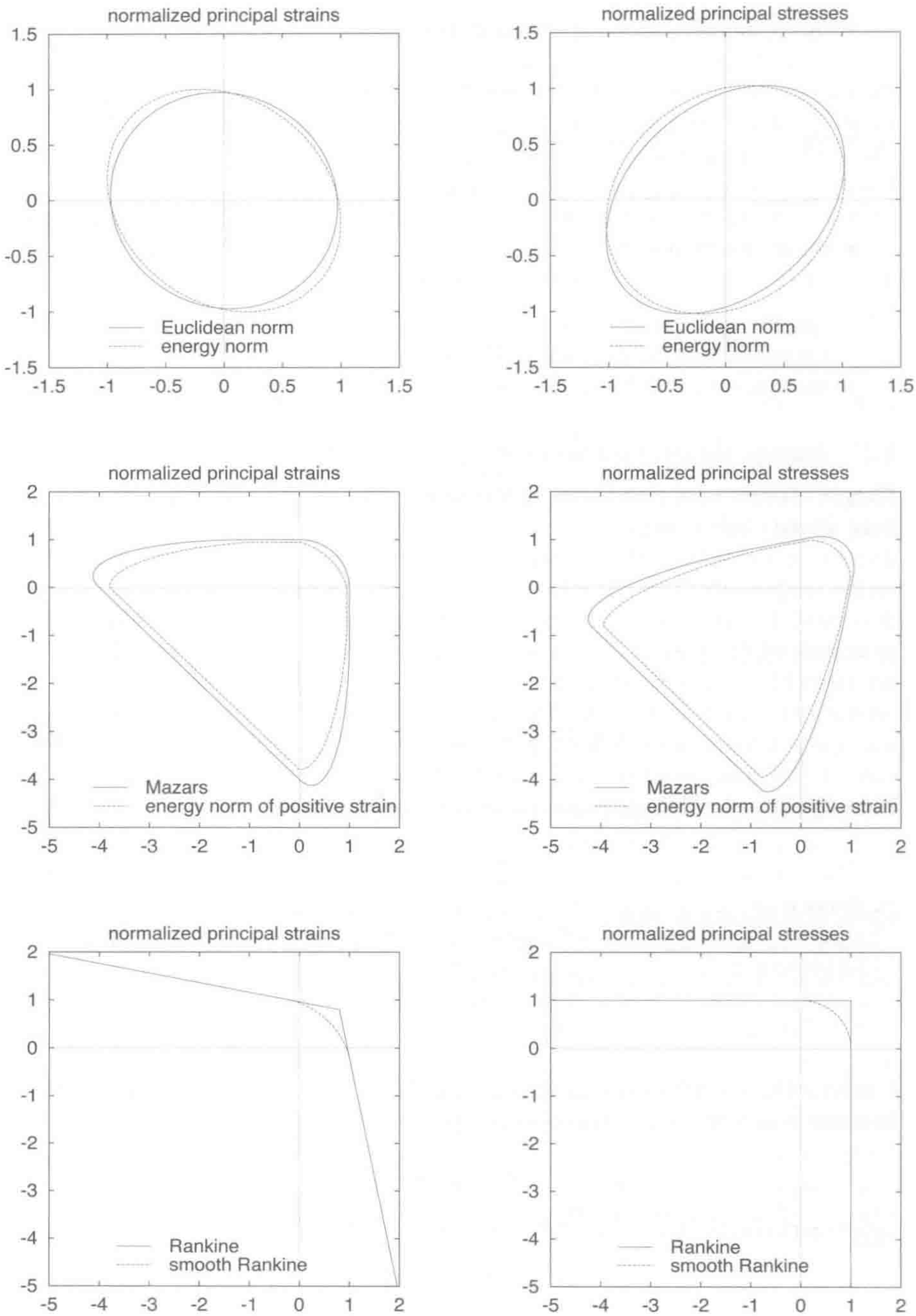

Figure 3. Loading surfaces for various definitions of equivalent strain

which is the multidimensional generalization of (1).

Similar to the uniaxial case, we introduce a loading function f specifying the elastic domain and the states at which damage grows. The loading function now depends on the strain tensor, ε, and on a variable κ that controls the evolution of the elastic domain. Physically, κ is a scalar measure of the largest strain level ever reached in the history of the material. States for which $f(\varepsilon, \kappa) < 0$ are supposed to be below the current damage threshold. Damage can grow only if the current state reaches the boundary of the elastic domain. This is described by the loading-unloading conditions (8). It is convenient to postulate the loading function in the form

$$f(\varepsilon, \kappa) = \tilde{\varepsilon}(\varepsilon) - \kappa \tag{17}$$

where $\tilde{\varepsilon}$ is the *equivalent strain*, i.e., a scalar measure of the strain level.

In some sense, the expression defining the equivalent strain plays a role similar to the yield function in plasticity, because it directly affects the shape of the elastic domain. The simplest choice is to define the equivalent strain as the Euclidean norm of the strain tensor,

$$\tilde{\varepsilon} = \|\varepsilon\| = \sqrt{\varepsilon : \varepsilon} = \sqrt{\varepsilon_{ij}\varepsilon_{ij}} \tag{18}$$

or as the energy norm,

$$\tilde{\varepsilon} = \sqrt{\frac{\varepsilon : \mathbb{E} : \varepsilon}{E}} = \sqrt{\frac{1}{E}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}} \tag{19}$$

where E_{ijkl} are the components of the elastic stiffness tensor $\mathbb{E}$ and normalization by Young's modulus E is introduced in order to obtain a strain-like quantity. Each particular definition of equivalent strain corresponds to a certain shape of the elastic domain in the strain space and can be transformed into the stress space. For illustration, Fig. 3(top) shows the elastic domains in projection onto the principal strain plane and in the principal stress plane for the case of plane stress and Poisson's ratio $\nu = 0.2$. The domains are elliptical and symmetric with respect to the origin. Consequently, there would be no difference in the response to tensile and compressive loadings.

For concrete and other materials with very different behaviors in tension and in compression, it is necessary to adjust the definition of equivalent strain. Microcracks in concrete grow mainly if the material is stretched, and so it is natural to take into account only normal strains that are positive and neglect those that are negative. This leads to the so-called Mazars definition of equivalent strain (Mazars, 1984)

$$\tilde{\varepsilon} = \|\langle\varepsilon\rangle\| = \sqrt{\langle\varepsilon\rangle : \langle\varepsilon\rangle} \tag{20}$$

or to its energetic counterpart,

$$\tilde{\varepsilon} = \sqrt{\frac{\langle \varepsilon \rangle : \mathbb{E} : \langle \varepsilon \rangle}{E}} \tag{21}$$

where McAuley brackets $\langle . \rangle$ denote the "positive part" operator. For scalars, $\langle x \rangle = \max(0, x)$, i.e., $\langle x \rangle = x$ for x positive and $\langle x \rangle = 0$ for x negative. For symmetric tensors, such as the strain tensor $\boldsymbol{\varepsilon}$, the positive part is a tensor having the same principal directions $\boldsymbol{n}_I$ as the original one, with principal values ε_I replaced by their positive parts $\langle \varepsilon_I \rangle$. The subscript I ranges from 1 to 3 (the number of spatial dimensions) but it is not subject to Einstein's summation convention because the principal strains ε_I are not components of a first-order tensor. In terms of the spectral decomposition

$$\boldsymbol{\varepsilon} = \sum_{I=1}^{3} \varepsilon_I \, \boldsymbol{n}_I \otimes \boldsymbol{n}_I \tag{22}$$

the positive part of ε is expressed as

$$\langle \boldsymbol{\varepsilon} \rangle = \sum_{I=1}^{3} \langle \varepsilon_I \rangle \, \boldsymbol{n}_I \otimes \boldsymbol{n}_I \tag{23}$$

Since $(\boldsymbol{n}_I \otimes \boldsymbol{n}_I) : (\boldsymbol{n}_J \otimes \boldsymbol{n}_J) = \delta_{IJ} =$ Kronecker's delta, definition (20) can be rewritten as

$$\tilde{\varepsilon} = \sqrt{\sum_{I=1}^{3} \langle \varepsilon_I \rangle^2} \tag{24}$$

The elastic domains corresponding to (20) and (21) are shown in Fig. 3(center).

If a model corresponding to the Rankine criterion of maximum principal stress is desired, one may use the definitions

$$\tilde{\varepsilon} = \frac{1}{E} \max_{I=1,2,3} \langle \mathbb{E} : \boldsymbol{\varepsilon} \rangle_I = \frac{1}{E} \max_{I=1,2,3} \langle \bar{\sigma}_I \rangle \tag{25}$$

or

$$\tilde{\varepsilon} = \frac{1}{E} \| \langle \mathbb{E} : \boldsymbol{\varepsilon} \rangle \| = \frac{1}{E} \sqrt{\sum_{I=1}^{3} \langle \mathbb{E} : \boldsymbol{\varepsilon} \rangle_I^2} = \frac{1}{E} \sqrt{\sum_{I=1}^{3} \langle \bar{\sigma}_I \rangle^2} \tag{26}$$

where $\langle \bar{\sigma}_I \rangle = \langle \mathbb{E} : \boldsymbol{\varepsilon} \rangle_I$, $I = 1, 2, 3$, are the positive parts of principal values of the effective stress tensor (15). The former definition exactly corresponds to the Rankine criterion while the latter rounds off the corners in the octants with more than one positive principal stress; see Fig. 3(bottom).

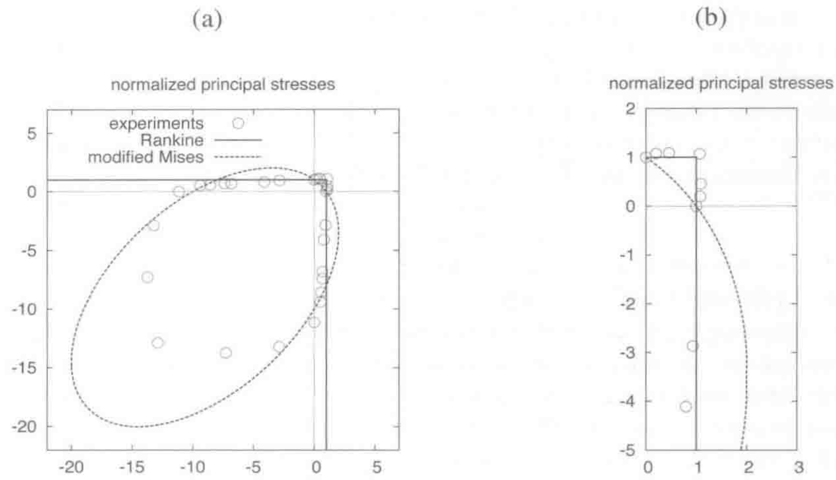

Figure 4. Biaxial strength envelope for concrete and its approximation by isotropic damage models with Rankine and modified Mises definition of equivalent strain

An alternative formula, called the modified von Mises definition (de Vree et al., 1995), reads

$$\tilde{\varepsilon} = \frac{(k-1)I_{1\varepsilon}}{2k(1-2\nu)} + \frac{1}{2k}\sqrt{\frac{(k-1)^2}{(1-2\nu)^2}I_{1\varepsilon}^2 + \frac{12kJ_{2\varepsilon}}{(1+\nu)^2}} \tag{27}$$

where

$$I_{1\varepsilon} = \mathbf{1} : \boldsymbol{\varepsilon} = 3\,\varepsilon_V \tag{28}$$

is the first strain invariant (trace of the strain tensor),

$$J_{2\varepsilon} = \tfrac{1}{2}\mathbf{e} : \mathbf{e} = \tfrac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} - \tfrac{1}{6}I_{1\varepsilon}^2 \tag{29}$$

is the second deviatoric strain invariant, and k is a model parameter that sets the ratio between the uniaxial compressive strength f_c and uniaxial tensile strength f_t. The elastic domains corresponding to the modified von Mises definition have ellipsoidal shapes but their centers are shifted from the origin along the hydrostatic axis (except for the special case with parameter $k = 1$, which corresponds to the standard von Mises definition, with equivalent strain proportional to $\sqrt{J_{2\varepsilon}}$).

The uniaxial tensile strength and uniaxial compressive strength can be fitted, but the shape of the elastic domain in the tension-compression quadrant of the principal stress plane does not correspond to experimental data for concrete (Kupfer et al., 1969) and the shear strength is overestimated, see Fig. 4.

An important advantage of isotropic damage models is that the stress evaluation algorithm is usually explicit, without the need for an iterative solution of one or more nonlinear equations. The choice of a loading function in the form (17) endows the variable κ with the meaning of the largest value of equivalent strain that has ever occured in the previous deformation history of the material up to its current state; cf. (8). In other words, (6) can be generalized to

$$\kappa(t) = \max_{\tau \leq t} \tilde{\varepsilon}(\tau) \tag{30}$$

For a prescribed strain increment, the corresponding stress is evaluated simply by computing the current value of equivalent strain, updating the maximum previously reached equivalent strain and the damage variable, and reducing the effective stress according to (14). Depending on the definition of equivalent strain one may have to extract the principal strains or principal stresses. This can be done very easily, since closed-form formulas for the eigenvalues of symmetric matrices of size 2×2 or 3×3 are available.

The damaged stiffness tensor $\mathbb{E}_S = (1 - D)\mathbb{E}$ introduced in (13) links the total stress to total strain and plays the role of the tangent stiffness only for unloading with constant damage ($f < 0$ or $\dot{f} < 0$). To construct the tangent stiffness tensor for loading with growing damage ($f = 0$ and $\dot{f} = 0$), we need to find the link between stress and strain increments or rates. The damage rate can be expressed in terms of the strain rate using the consistency condition $\dot{f} = 0$ with the rate of the damage loading function evaluated from (17) and combining it with the rate form of equation (11):

$$\dot{D} = \frac{\mathrm{d}g}{\mathrm{d}\kappa}\dot{\kappa} = \frac{\mathrm{d}g}{\mathrm{d}\kappa}\dot{\tilde{\varepsilon}} = \frac{\mathrm{d}g}{\mathrm{d}\kappa}\frac{\partial\tilde{\varepsilon}}{\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \tag{31}$$

For convenience, we introduce symbols g' for the derivative $\mathrm{d}g/\mathrm{d}\kappa$ of the damage function, and $\boldsymbol{\eta}$ for the second order tensor $\partial\tilde{\varepsilon}/\partial\boldsymbol{\varepsilon}$ obtained by differentiation of the expression for the equivalent strain with respect to the strain tensor. Substituting $\dot{D} = g'\boldsymbol{\eta} : \dot{\boldsymbol{\varepsilon}}$ into the rate form of the stress-strain law (14) we get

$$\dot{\boldsymbol{\sigma}} = (1 - D)\mathbb{E} : \dot{\boldsymbol{\varepsilon}} - \mathbb{E} : \boldsymbol{\varepsilon}\dot{D} = (1 - D)\mathbb{E} : \dot{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}\,(g'\boldsymbol{\eta} : \dot{\boldsymbol{\varepsilon}}) = \mathbb{E}_{ed} : \dot{\boldsymbol{\varepsilon}} \tag{32}$$

where $\bar{\boldsymbol{\sigma}} = \mathbb{E} : \boldsymbol{\varepsilon}$ is the effective stress and

$$\mathbb{E}_{ed} = (1 - D)\mathbb{E} - g'\bar{\boldsymbol{\sigma}} \otimes \boldsymbol{\eta} \tag{33}$$

is the elasto-damage stiffness tensor. It is interesting to note that for a model with the equivalent strain based on the energy norm, eq. (19), the tensor $\boldsymbol{\eta}$ is given by

$$\boldsymbol{\eta} = \frac{\partial\tilde{\varepsilon}}{\partial\boldsymbol{\varepsilon}} = \frac{1}{2\sqrt{\dfrac{\boldsymbol{\varepsilon} : \mathbb{E} : \boldsymbol{\varepsilon}}{E}}}\frac{1}{E}2\mathbb{E} : \boldsymbol{\varepsilon} = \frac{\bar{\boldsymbol{\sigma}}}{E\tilde{\varepsilon}} \tag{34}$$

and the resulting elasto-damage stiffness tensor

$$\mathbb{E}_{ed} = (1 - D)\mathbb{E} - \frac{g'}{E\tilde{\varepsilon}}\bar{\boldsymbol{\sigma}} \otimes \bar{\boldsymbol{\sigma}} \tag{35}$$

exhibits major symmetry ($E^{\text{ed}}_{ijkl} = E^{\text{ed}}_{klij}$). For other definitions of equivalent strain, this kind of symmetry is lost.

Mazars Damage Model. A popular damage model specifically designed for concrete was proposed by Mazars (Mazars, 1984, 1986). He introduced two damage variables, D_t and D_c, that are computed from the same equivalent strain (24) using two different damage functions, g_t and g_c. Function g_t is identified from the uniaxial tensile test while g_c corresponds to the compressive test. The damage variable entering the constitutive equations (14) is $D = D_t$ under tension and $D = D_c$ under compression. For general stress states the value of D is obtained as a linear combination

$$D = \alpha_t D_t + \alpha_c D_c \tag{36}$$

where the coefficients α_t and α_c take into account the character of the stress state. In the recent implementation of Mazars model, these coefficients are evaluated as

$$\alpha_t = \left(\sum_{I=1}^{3} \frac{\varepsilon_{tI}\langle\varepsilon_I\rangle}{\tilde{\varepsilon}^2}\right)^{\beta}, \qquad \alpha_c = \left(1 - \sum_{I=1}^{3} \frac{\varepsilon_{tI}\langle\varepsilon_I\rangle}{\tilde{\varepsilon}^2}\right)^{\beta} \tag{37}$$

where ε_{tI}, $I = 1, 2, 3$, are the principal strains due to positive stresses, i.e., the principal values of $\boldsymbol{\varepsilon}_t = \mathbb{C} : \langle \mathbb{E} : \boldsymbol{\varepsilon}\rangle$, in which $\mathbb{C} = \mathbb{E}^{-1}$ is the elastic compliance tensor. The exponent $\beta = 1.06$ slows down the evolution of damage under shear loading (i.e., when principal stresses do not have the same sign). In the original version of the model (Mazars, 1984), β was equal to 1.

Note that if all principal stresses are nonnegative we have $\alpha_t = 1$, $\alpha_c = 0$, and $D = D_t$, and if all principal stresses are nonpositive we have $\alpha_t = 0$, $\alpha_c = 1$, and $D = D_c$. These are the "purely tensile" and "purely compressive" stress states. For intermediate stress states the value of D is between D_t and D_c, depending on the relative magnitudes of tensile and compressive stresses. Functions characterizing the evolution of damage were originally proposed in the form (Mazars, 1984)

$$g_t(\kappa) = \begin{cases} 0 & \text{if} \quad \kappa \le \varepsilon_0 \\ 1 - (1 - A_t)\dfrac{\varepsilon_0}{\kappa} - A_t \exp\left[-B_t(\kappa - \varepsilon_0)\right] & \text{if} \quad \kappa \ge \varepsilon_0 \end{cases} \tag{38}$$

$$g_c(\kappa) = \begin{cases} 0 & \text{if} \quad \kappa \le \varepsilon_0 \\ 1 - (1 - A_c)\dfrac{\varepsilon_0}{\kappa} - A_c \exp\left[-B_c(\kappa - \varepsilon_0)\right] & \text{if} \quad \kappa \ge \varepsilon_0 \end{cases} \tag{39}$$