INTRODUCTORY COLLEGE MATHEMATICS

HACKWORTH
and
HOWLAND

SAUNDERS SERIES IN MODULAR MATHEMATICS

Linear Programming

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PREFACE

Linear Programming

This book is one of the sixteen content modules in the Saunders Series in Modular Mathematics. The modules can be divided into three levels, the first of which requires only a working knowledge of arithmetic. The second level needs some elementary skills of algebra and the third level, knowledge comparable to the first two levels. *Linear Programming* is in level 3. The groupings according to difficulty are shown below.

Level 1	Level 2	Level 3
Tables and Graphs	*Numeration*	*Real Number System*
Consumer Mathematics	*Metric Measure*	*History of Real Numbers*
Algebra 1	*Probability*	*Indirect Measurement*
Sets and Logic	*Statistics*	*Algebra 2*
Geometry	*Geometric Measures*	*Computers*
		Linear Programming

The modules have been class tested in a variety of situations: large and small discussion groups, lecture classes, and in individualized study programs. The emphasis of all modules is upon ideas and concepts.

Linear Programming is appropriate for all non-science majors and is especially relevant for business and well prepared liberal arts students. The module will also be useful for mathematics majors.

Linear Programming begins by presenting methods of graphing optimum situations involving two factors. After the student has achieved a visual understanding through graphing, methods of maximizing or minimizing production problems are developed. *Linear Programming* includes a presentation of the simplex method and its application to practical situations.

In preparing each module we have been greatly aided by the valuable suggestions of the following excellent reviewers: William Andrews, Triton College, Ken Goldstein, Miami-Dade Community College, Don Hostetler, Mesa Community College, Karl Klee, Queensboro Community College, Pamela Matthews, Chabot College, Robert Nowlan, Southern Connecticut State College, Ken Seydel, Skyline College, Ara Sullenberger, Tarrant County Junior College, and Ruth Wing, Palm Beach Junior College. We thank them, and the staff at W. B. Saunders Company for their support.

Robert D. Hackworth
Joseph W. Howland

NOTE TO THE STUDENT

OBJECTIVES:

Upon completing this unit the reader is expected to be able to demonstrate the following skills and concepts:

1. Finding the solution set of two inequalities by graphing.

2. Solving linear programming problems involving two variables using graphing methods.

3. Solving linear programming problems with three or more variables by using the simplex method.

Three types of problem sets with answers are included in this module. Progress Tests appear at the end of each section. These Progress Tests are always short with only four to six problems. The questions asked in Progress Tests always come directly from the material of the section immediately preceding the test.

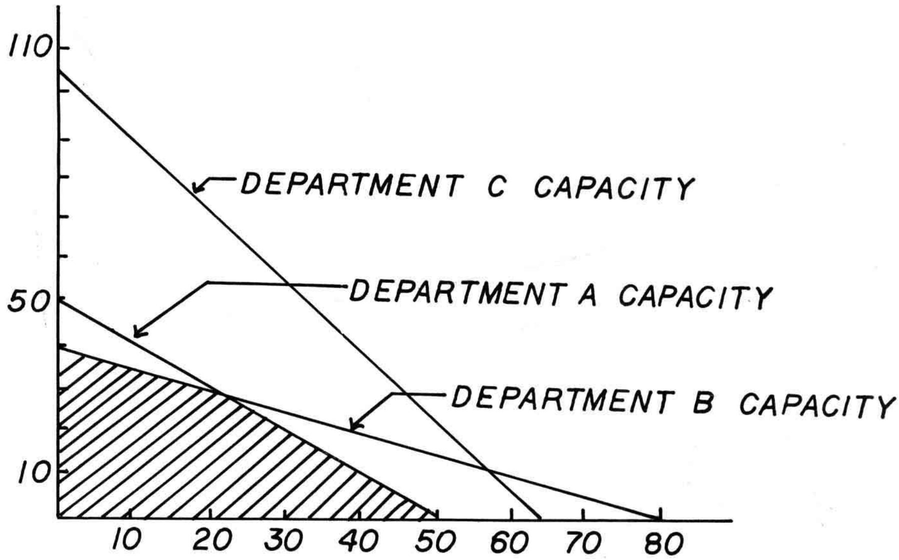
Exercise Sets appear less frequently in the module. More problems appear in an Exercise Set than in a Progress Test and these problems arise from all sections of the module preceding the Exercise Set. Problems in the Exercise Sets are specifically chosen to match the objectives of the module.

A Self-Test is found at the end of the module. Self-Tests contain problems representative of the entire module.

In learning the material, the student is encouraged to try each problem set as it is encountered, check all answers, and restudy those sections where difficulties are discovered. This procedure is guaranteed to be both efficient and effective.

AJAX COMPANY RECOURCES

DEPARTMENT	MINUTES REQUIRED PER UNIT: PRODUCT A	MINUTES REQUIRED PER UNIT: PRODUCT B	CAPACITY PER DAY IN MINUTES
A	6	6	300
B	4	8	320
C	5	3	310
PROFIT CONTRIBUTION PER UNIT	$10	$12	



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LINEAR PROGRAMMING

"MORE" IS NOT NECESSARILY "BETTER"

The story is told of the doctor who recommended that his patient, suffering from poor blood circulation, have a cocktail before dinner each evening. The patient, being truly interested in his health, had two or three cocktails each evening so that he would be two or three times as healthy. The patient died of liver malfunctioning.

The moral of the story is that one cocktail was the optimum amount for the patient and more or less than one cocktail per day was not beneficial.

Many life situations require finding an optimum number or range of numbers. Numbers greater or less than the optimum should be avoided.

Other examples of situations where an optimum number or range of numbers is appropriate are:

1. Each car has a range of speeds in which the motor works most efficiently and gas use is minimized.

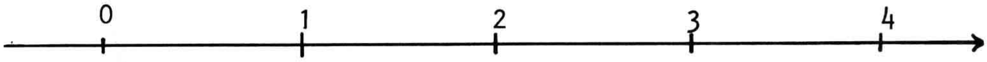
2. Each individual has a range of hours which should be slept each night. Sleeping more or less hours is generally unwise and/or unnecessary.

3. The caloric intake of an individual each day has an optimum range.

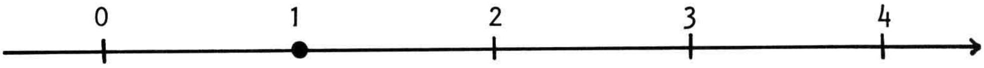
4. The number of aspirins suggested by most manufacturers is one or two.

5. Some critics of the population explosion suggest that the optimum range of average childbirths per woman is zero to 2.

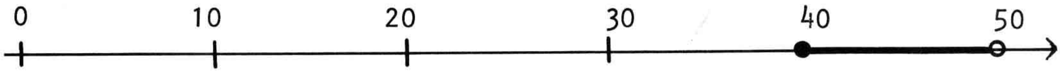
Each of these situations can be shown graphically using a number line like the one shown below.



The blood circulation patient can have his optimum number of cocktails per day graphed by placing a darkened circle at the number 1. His optimum use of cocktails per day is shown below.



The range of speeds in which a car operates most efficiently is graphed by darkening a portion of the number line. If a particular car has an optimum speed range of at least 40 m.p.h. but less than 50 m.p.h., the graph would appear as shown below.



The dot at 40 has been darkened to indicate that 40 m.p.h. is an acceptable number in the range. The dot at 50 has not been darkened to indicate that the optimum range is "less than 50 m.p.h." and 50 is not included.

Progress Test 1

Draw a number line graph for each of the following "optimum" situations:

1. John Jones was advised to get at least 7 hours and less than 10 hours sleep each night.

2. Henry Jones was advised to keep his caloric intake each day greater than 800 and less than 2400.

3. The number of aspirins recommended for headache relief is 1 or 2. Jim Jones has a headache.

4. The desired average number of childbirths per woman is 0.25 to 2.

OPTIMUM PROBLEMS INVOLVING TWO VARIABLES

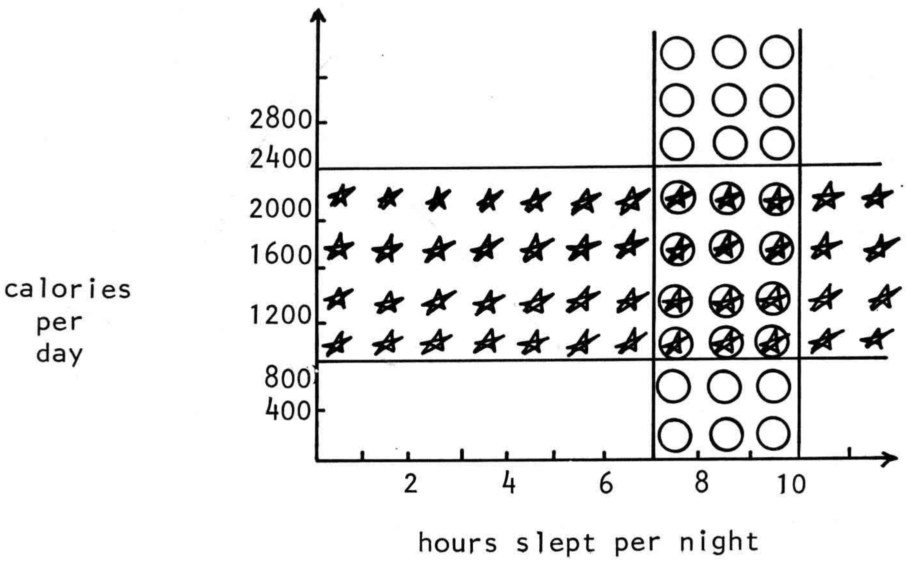
Each problem of the preceding section concentrated upon a single factor which determined the optimum situation. Generally speaking, a person's health or the gas mileage achieved by a car are situations which are dependent upon more than one factor. In this section of the module the problems are made slightly more complex by involving two factors or variables.

The health of the blood circulation patient could be viewed as dependent upon both the number of hours slept each night and the number of calories per day. If so, the optimum conditions for good health may involve

1. At least 7 hours and less than 10 hours sleep each night.

2. A caloric intake per day greater than 800 and less than 2400.

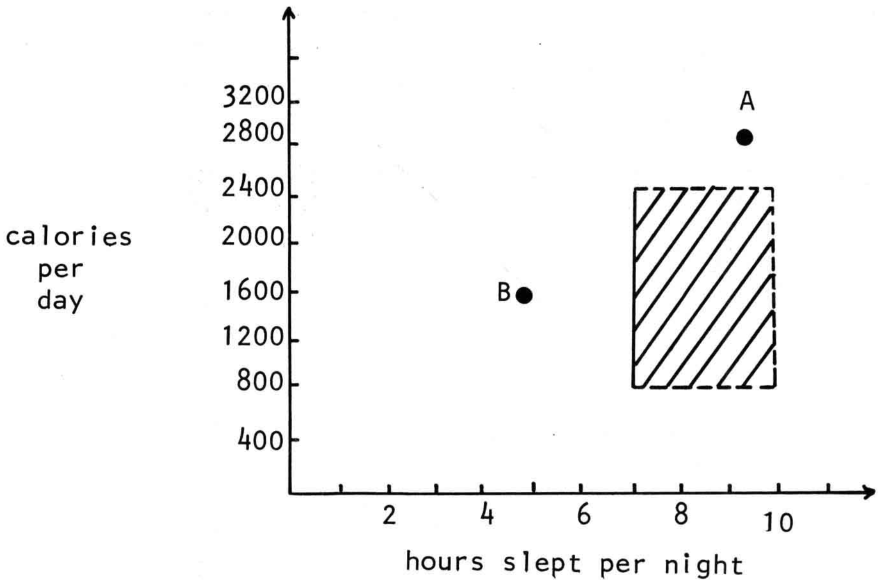
These optimum conditions may be graphed using two perpendicular axes as shown below.



The horizontal axis is labeled "hours slept per night". The vertical column filled with circles represents that area of the graph in which the hours slept is "at least 7 hours and less than 10 hours".

The vertical axis is labeled "calories per day". The horizontal row filled with stars represents the area of the graph in which the calories are "greater than 800 and less than 2400".

The area of optimum health, with respect to the two factors, is the area marked by both circles and stars. The optimum health graph is shown below.



The graph shows a cross-hatched rectangle as the optimum health area. Three sides of the rectangle are dotted lines meaning that a point on the line is not included in the optimum range. One side of the rectangle is a solid line meaning that points on the line are included in the optimum range.

Two points A and B are indicated on the graph. Point A is not in the optimum area because its "calorie" level is 2800, which exceeds the optimum range for that variable. Point B is not in the optimum area because its "hours slept" representation is 5 which is less than the optimum range for that variable.

Progress Test 2

1. On a horizontal axis graph the optimum range for: An automobile achieves its best gas mileage at a speed of at least 40 m.p.h. but less than 50.

2. On a vertical axis graph the optimum range for: An automobile achieves its best gas mileage after it is "broken in" at 5000 miles and before it reaches 50,000 miles.

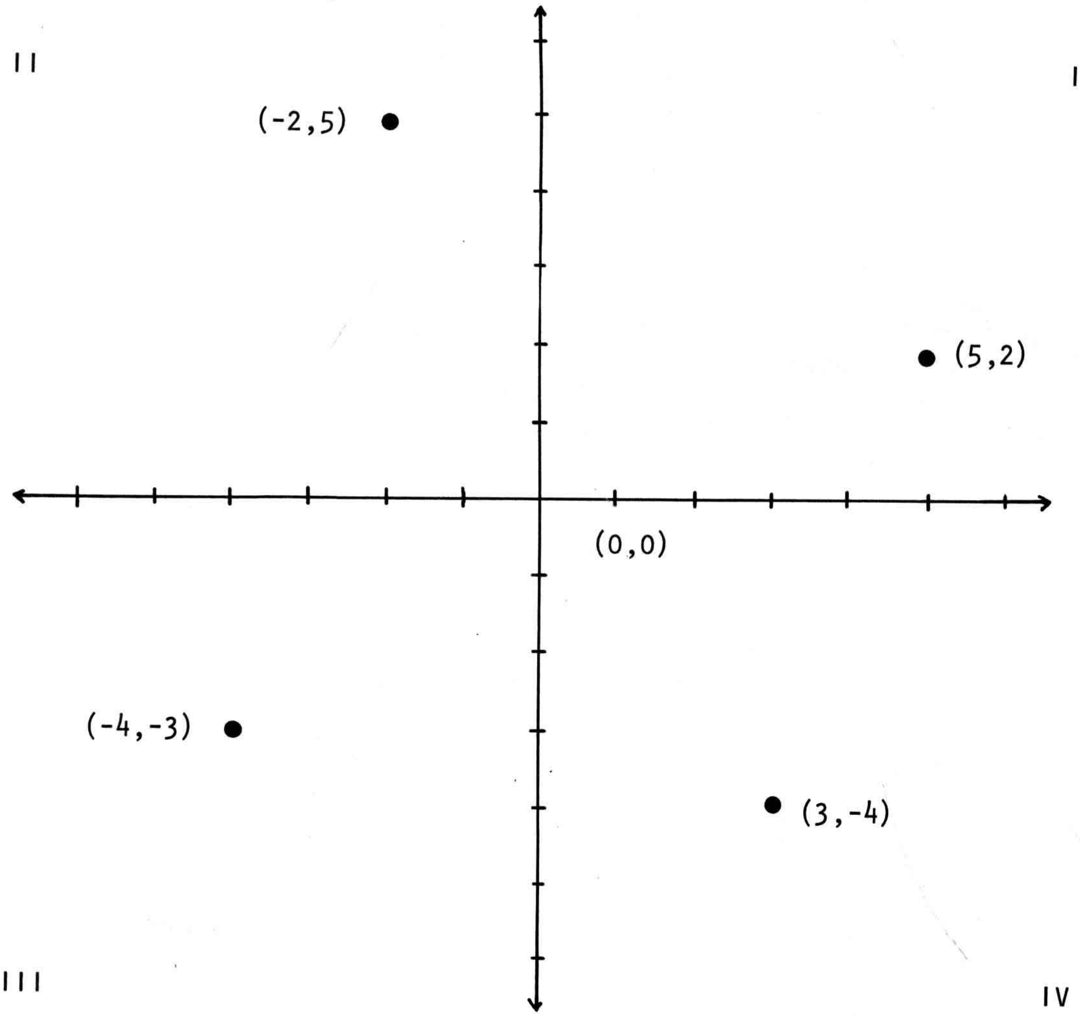
3. Make columns of circles showing the optimum gas mileage with respect to speed.

4. Make rows of stars to show the optimum gas mileage with respect to the number of miles driven.

5. Show by a cross-hatched area the optimum gas mileage with respect to the two variables.

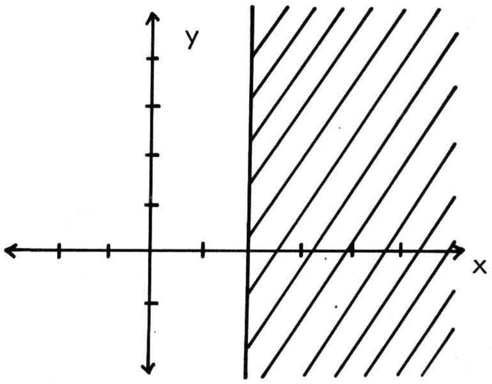
USING LETTERS TO REPRESENT THE VARIABLES

In the last section the example used two factors stated in words. Oftentimes the factors in such examples are represented by letters or variables. The graph below shows two intersecting axes. The horizontal line is labeled x and is called the x-axis. The vertical line is labeled y and is called the y-axis.

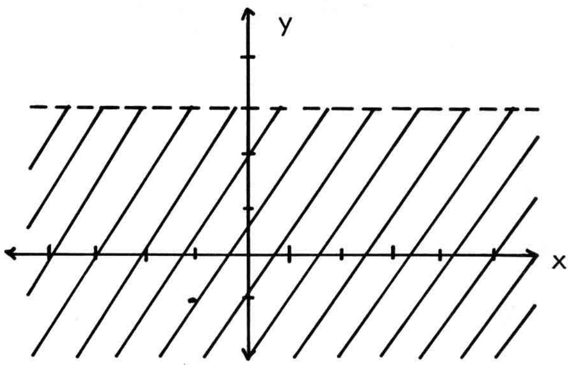


The lines separate the plane into four parts indicated by the Roman numerals I, II, III, and IV. Each point of the plane is represented by an ordered pair (x,y) where the x number represents the directed distance from the y-axis and the y number the directed distance from the x-axis. The ordered pair (0,0) names the point where the axes intersect. Four other points (ordered pairs) are indicated on the graph. In each case the first number in the ordered pair is the x component and the second number in the ordered pair is the y component.

The symbol "$\geq$" is read as "is greater than or equal to." The graph of the inequality $x \geq 2$ consists of all those points (ordered pairs) where the first number is greater than or equal to 2. The cross-hatched area of the figure shows the graph of $x \geq 2$. The boundary line of the cross-hatched area is a solid line to indicate that x can be equal to 2.



The symbol "<" is read as "is less than." The graph of the inequality $y < 3$ consists of all points (ordered pairs) where the second component is less than 3. The cross-hatched area shows the graph of $y < 3$. The boundary line of the cross-hatched area is a dotted line to show that y cannot be equal to 3.

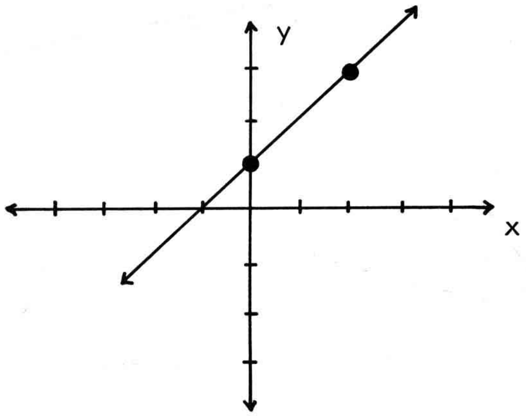


To graph the inequality $y \geq x + 1$, first graph $y = x + 1$ by using any two ordered pairs (x,y) that make $y = x + 1$ a true statement. To do this choose any number whatsoever to replace x and determine its matching y number.

If $x = 0$, then $y = x + 1 = 0 + 1 = 1$ and $(0,1)$ is a solution.

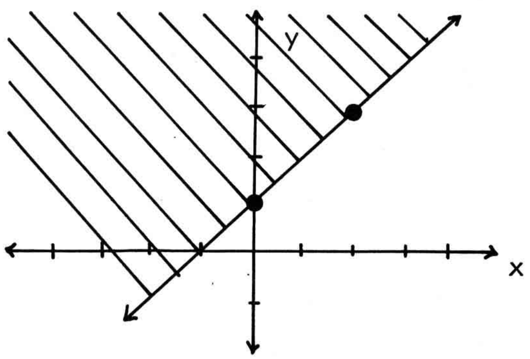
If $x = 2$, then $y = x + 1 = 2 + 1 = 3$ and $(2,3)$ is a solution.

Using $(0,1)$ and $(2,3)$ the graph of $y = x + 1$ is a straight line drawn through the two points.



The graph of $y > x + 1$ is either the half-plane above or below the line of $y = x + 1$. Since $(0,0)$ is not on the line $y = x + 1$, $(0,0)$ either is or is not a solution of $y > x + 1$.

$0 > 0 + 1$ is a false statement. Therefore, $(0,0)$ is not a solution of $y > x + 1$. This indicates that the graph of $y > x + 1$ is the area above the line of $y = x + 1$. The completed graph of $y \geq x + 1$ is shown below by the shaded portion of the graph.



Progress Test 3

Graph each of the following inequalities on an (x,y) plane.

1. $x \geq 5$

2. $y < 1$

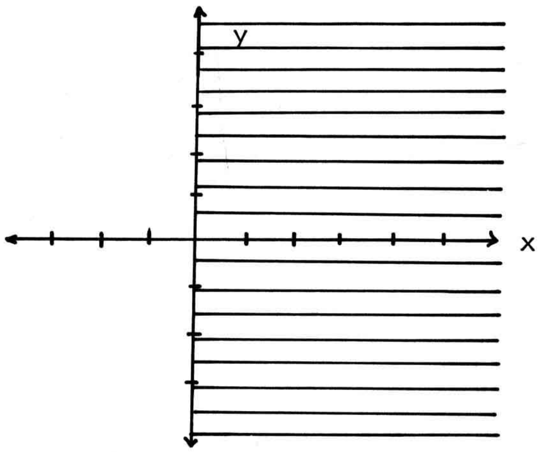
3. $x + y < 7$

4. $2x + 3y \geq 6$

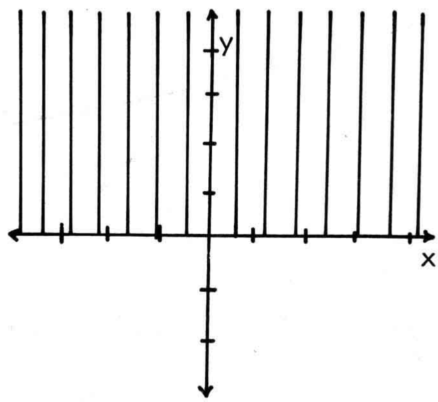
The ability to graph inequalities or constraints is important in locating optimum solutions for problems involving two factors. This is because the factors can be represented by the variables x and y. The intersection of the graphs of the separate inequalities will represent the acceptable solutions to the problem.

The following steps are involved in graphing solutions for the constraint $x \geq 0$, $y \geq 0$, and $x + y \leq 6$.

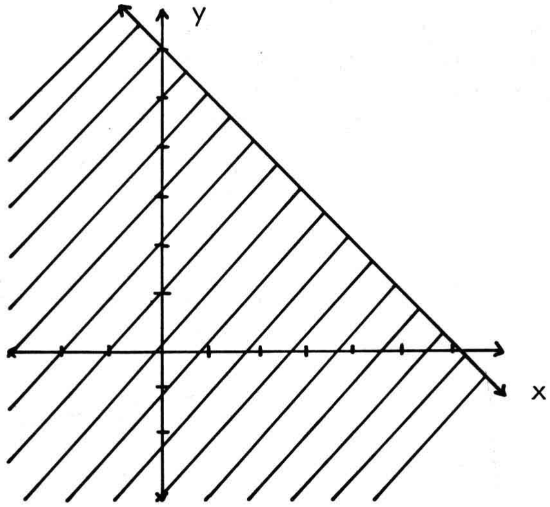
Each inequality is graphed separately.



$x \geq 0$



$y \geq 0$



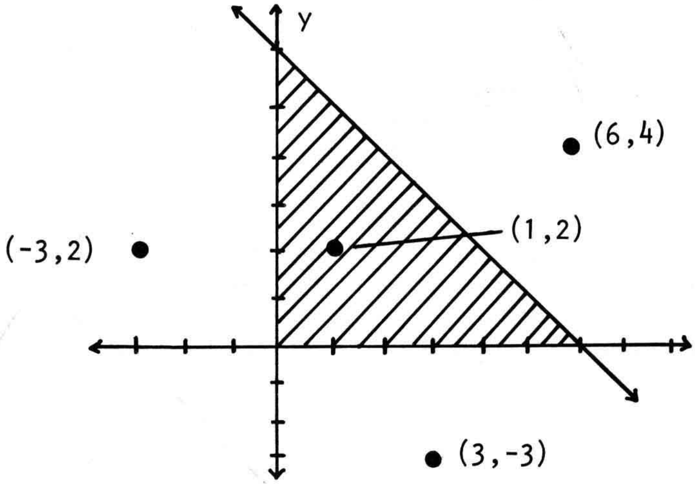
$x + y \leq 6$

The three graphs are overlapped and only those points covered by all three areas are accepted.

$x \geq 0$

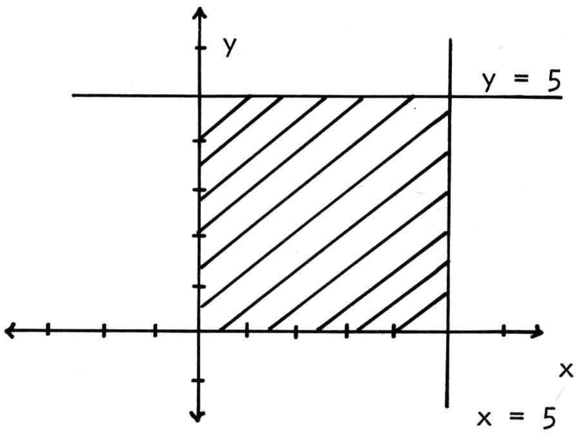
$y \geq 0$

$x + y \leq 6$



For ordered pairs in the shaded area above, the first components are greater than or equal to zero, the second components are greater than or equal to zero, and the sum of the two components is less than or equal to six. Consequently, (6,4), (-3,2), and (3,-3) are not in the shaded area of the graph above. (1,2) is in the shaded area above because its components conform to all of the constraints.

To graph the ordered pairs that conform to the restrictions, $x \geq 0$, $y \geq 0$, $x \leq 5$, and $y \leq 5$, first graph each of the four inequalities and shade the area where the four graphs overlap. Different colors can be used for each inequality to emphasize the intersection area. The overlapped area will represent the intersection of the four solution sets. The graph will consist of all the ordered pairs where the first component is greater than or equal to zero and less than or equal to five. Also, the second component will be greater than or equal to zero and less than or equal to five. The graph is shown below.



The first step in graphing the ordered pairs that meet the constraints $x \geq 0$, $y \geq 0$, $3x + y \geq 12$ and $x + 2y \geq 8$ is to find two ordered pair solutions for each of the equations $3x + y = 12$ and $x + 2y = 8$. For $3x + y = 12$, if $x = 0$, then $y = 12$. If $x = 2$ then $3x + y = 12$ becomes $3 \cdot 2 + y = 12$ and $y = 6$. Therefore, the ordered pairs (0,12) and (2,6) are solutions of $3x + y = 12$.

For $x + 2y = 8$, if $x = 0$, then $y = 4$. If $x = 4$, then $x + 2y = 8$ becomes $4 + 2y = 8$ and $y = 2$. Therefore, the ordered pairs (0,4) and (4,2) are solutions of $x + 2y = 8$. The graph below shows the ordered pairs that meet the constraints above.