Nonlinear Dynamics

A TWO WAY TRIP FROM PHYSICS TO MATH

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There are more things in heaven and earth, Horatio
Than are dreamt of in your philosophy...
W Shakespeare, Hamlet, Act 1, Scene 5.

To:
Bárbara, Patrizia and Silvia
Florencia and Julia
and **Cristal**

Acknowledgments

... si acaso fuera a quedar de mis deudas un haber...

José Larralde, Herencia pa' un hijo gaucho

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Preface

About the book

The aim of this book is to render available to readers the tools that nonlinear dynamics provides for the exploration of new problems in all fields of physics. In research we deal with open problems: problems for which, at the beginning, we have no solutions but at most a set of hunches, feelings and guesses based on our previous experience with other problems. From there, we work our path to the solutions (though we do not always succeed). In finding our way we take what seems to us the *most natural* approach just as we would bush-walk in the forest avoiding as much as possible the difficult paths in our hike towards an interesting place.

We will follow in the presentation the same procedure that we follow when doing research, i.e., we begin with the problem and find one path to the solution, we work inductively proposing new paths, checking them and redrawing our route according to the experience we gain in successive efforts.

We will avoid the temptation of selecting the problems according to the tools we possess. Rather, we prefer to construct the tools along with the problems. We sustain the idea that our tools (theories) and our problems evolve hand in hand. The best pages of physics have been written in this way. Consider for example the pairs calculus–mechanics (Newton) or Hilbert spaces–quantum theory (Von Neumann) and, as we will see, nonlinear dynamics–topology (Poincaré).

This is a book to be read with paper and pencil at hand. Our intention is to furnish readers with enough knowledge to be able to do research in nonlinear dynamics after having read the book (or better, *while* they are reading the book). We prefer to convey the key ideas within their mathematical framework rather than doing lengthy demonstrations. Therefore, the calculations around the results presented in the book are usually only sketched or left more or less as guided exercises.

At the end of the day, we would like the reader to finish this book with the feeling (or certainty) that, given enough time, she/he would have come up with the same answers to the problems as those we have shown (well ... perhaps just better answers). After all *the answers are dictated by the problems*, the two of them evolve in interaction, and our task is to read them from nature.

About nonlinear dynamics

Nonlinear dynamics is a subject at least as old in physics as Newton's mechanics. The dynamics of the planetary system, a primary concern for Newton, Poincaré and many others, turns out to be nonlinear in general, but fortunately the simplest examples can be solved exactly (two-body systems are completely integrable). In the context of Hamiltonian mechanics a completely integrable system is 'almost' a linear system in an appropriate set of coordinates (those given by the Hamilton–Jacobi theory for integrable systems [arno89]).

In contrast, three-body problems are also nonlinear, but in general very complex and non-integrable. It was while studying the three-body problem that Poincaré gave a new and important impulse to nonlinear dynamics at the beginning of the 20th century. However, the new physics of the atom (later the nucleus, then quarks, ...) caught the attention of physicists. There was apparently no use for nonlinear dynamics in quantum mechanics since the latter rests on Hilbert spaces (linear spaces after all). The excess of zeal with quantum mechanics caused the (almost complete) disappearance of nonlinear dynamics from physics and especially from textbooks.

The emergence of computers as new tools for theoretical physics in the late 1950s and early 1960s favoured a comeback of nonlinear dynamics. Numerical simulations made accessible to the intuition of physicists and non-physicists the richness of nonlinear models. The graphical output added an artistic touch.

Commensurate with its earlier neglect, the impact of the phenomenology of nonlinear dynamics rocked the physics community in the early 1980s, to the point that people even talked of a 'new science' [glei87].

Today, we have a calmer perspective. We recognize that many situations can only be described with nonlinear interactions. There is a growing consciousness that the tools, methods and phenomenology of nonlinear dynamics will be increasingly necessary for the study of most subjects in physics and natural science in general.

The present text is far from being a complete guide to nonlinear dynamics but it covers the basic ideas for general systems. Some topics, though important for historical and even practical reasons for physicists, like one-dimensional maps and Hamiltonian mechanics, have not been emphasized, *on purpose*. We are certain that if we were to stress these special (singular) cases we would induce the wrong generalization, just as our generation was induced to think that Hamilton–Jacobi theory applied to all systems.

The discussion is presented in most cases having in mind low-dimensional systems, i.e., systems where the spatial aspects behave coherently. In general, spatio-temporal dynamics is known to a lesser degree than low-dimensional dynamics and the authors' knowledge of the subject is correspondingly more limited.

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Chapter 1

Nonlinear dynamics in nature

'man never regards what he possesses as so much his own, as what he does; and the labourer who tends a garden is perhaps in a truer sense its owner, than the listless voluptuary who enjoys its fruits... In view of this consideration, it seems as if all peasants and craftsmen might be elevated into artists; that is, men who love their labour for its own sake, improve it by their own plastic genius and inventive skill, and thereby cultivate their intellect, ennoble their character, and exalt and refine their pleasures. And so humanity would be ennobled by the very things which now, though beautiful in themselves, so often serve to degrade it... But, still freedom is undoubtedly the indispensable condition, without which even the pursuits most congenial to individual human nature can never succeed in producing such salutary influences. Whatever does not spring from a man's free choice, or is only the result of instruction and guidance, does not enter into his very being, but remains alien to his true nature; he does not perform it with truly human energies, but merely with mechanical exactness... we admire what he does, but despise what he is.'

Wilhelm Von Humboldt [chom87]

A sound theory starts usually with experiment. The cornerstone of natural sciences is precisely its validation procedure, namely the fact that relevant assertions can be put to test by way of suitable experiments and eventually be rejected (if the experiment *proves* them wrong). Consequently, the aim of this chapter is to present nonlinear problems from the experimental point of view. By *nonlinear* we just mean systems that demand to be modelled by nonlinear differential equations or nonlinear discrete-time mappings.

We will review some experiments that illustrate the kind of phenomenon we expect to find in nonlinear systems. The choice of examples is unavoidably biased and incomplete. We combine everyday experiences with well known problems, all of them sharing the characteristic features of nonlinearity. We want to emphasize that for the physicist nonlinear dynamics is far more than a theoretical entertainment. Moreover, we hope to make clear that *almost everything in nature is nonlinear.*

1.1 Hiking among rabbits

A few years ago one of us went to a National Park near Ushuaia, Argentina on a 2 week hiking trip. The park is placed in the southernmost part of *Tierra del Fuego*, the mysterious island that enchanted sailors, poets and natural scientists [darw88]. At that time, it was striking to notice that there were plenty of rabbits, and that many of them looked rather ill.

In fact, rabbits are not 'original' to the place, but were introduced (from Europe) by sailors as a way of having fresh meat when the ports of Tierra del Fuego were a natural stop in the route from the Atlantic to the Pacific, before the opening of the Panama Canal. Rabbits reproduced and spread all around the region (which suggests that they encountered no natural predators).

Towards the end of the 19th century the region was substantially occupied by cattle-farmers. Rabbits entered into competition with the cattle (sheep) and the farmers searched for a way of getting rid of them.

The farmers learned that a similar situation had arisen in Australia, where the matter was 'solved' by inoculating rabbits with a specific virus, which provoked a dramatic decrease in their population. The same approach was tried in Tierra del Fuego despite the opposition of the Park authorities and rangers. The inoculation was not as massive as in Australia, since the Park became virtually a protection sanctuary and the farmers did not find collaboration for their project.

After a sharp drop in population, rabbits in Australia recovered their previous population levels, developing a virus-resistant breed. In Argentina, after an initial decrease in population, a fluctuating state has been reached. There is an apparently periodic alternation between years with high levels of mortality and sickness and years with low levels of mortality.

The tale has many morals. However, we will concentrate on some physically relevant consequences, although they may not be the most important. To begin with we shall assume that the comparison between the Australian and the Argentine 'experiments' is scientifically valid.

To assess which are the reasons behind oscillatory behaviour in population dynamics is an interesting topic of research in itself. For our purposes, one would expect that periodic changes in rabbit populations would have a 'natural' period such as that of the season cycle or the reproduction cycle.

We begin noticing that *seemingly similar systems (rabbit populations) starting from apparently different initial states (the ratio between infected and healthy rabbits) can lead to the occurrence of different final (asymptotic) states.* This result may not be surprising recalling better known problems such as the ideal pendulum or a two-body Kepler problem.

The dramatically new result is the existence of oscillating cycles which extend over several years. What is surprising is not the periodic character of the solution but rather the fact that the period is *not* the 'natural' one. A recurrence of many times a given period is called *subharmonic behaviour*. Such behaviour cannot be achieved within a linear model.

Population dynamics, epidemiology and mathematical biology in general count among the most important contributors to the 'comeback' experienced by nonlinear science in recent decades. For example, very important names associated with nonlinear dynamics such as S Smale [smal76] and R May [may 75] have worked on this subject. A discussion concerning the example of the rabbits can be found in [dwye90]. Needless to say, the example of the rabbits is far from being unique.

1.2 Turbulence

The 1883 paper by Osborne Reynolds [reyn83] established the law of similarity that now bears his name (recall the *Reynolds number*). The article was concerned with two issues, one practical and one philosophical: 'the law of resistance of the motion of water in pipes' and 'that the general character of the motion of fluids in contact with solid surfaces depends on the relation between a physical constant of the fluid and the product of the linear dimensions of the space occupied by the fluid and the velocity'. Discussing these issues, Reynolds gave us the first study of the transition from laminar motion to turbulent motion in a fluid (*direct* and *sinuous* motion, as they are called in the paper). The exposition hardly needs further comments.

Reynolds states that 'The internal motion of water assumes one or another of two broadly distinguishable forms — either the elements of the fluid follow one another along the lines of motion which lead in the most direct manner to their destination, or they eddy about in sinuous paths the most indirect possible'. These are the laminar and turbulent forms of motion.

Further, we read that 'Certain circumstances have been definitely associated with the particular laws of force. Resistance, [varying] as the square of the velocity, is associated with motion in tubes of more than capillary dimensions, and with the motion of bodies through the water at more than insensibly small velocities, while resistance [varying] as the velocity is associated with capillary tubes and small velocities'. This is illustrated by figure 1.1 taken from Prandtl and Tietjens [pran34].

There are other circumstances that distinguish turbulent motion from laminar motion. For example, while laminar motion does not mix different stream lines, turbulent motion does. Therefore, if we colour one of the streamlines (see figure 1.2 taken from [reyn83]), or heat it up as in [barn04], only this streamline will be altered while, by contrast, in turbulent motion the whole fluid downstream of the alteration will be changed. Reynolds' description of this experiment is an excellent piece of scientific literature.

Having established that the transition between laminar and turbulent flow