AN INTRODUCTION TO THE FINITE ELEMENT METHOD WITH APPLICATIONS TO NONLINEAR PROBLEMS

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A Wiley-Interscience Publication
JOHN WILEY & SONS
New York • Chichester • Brisbane • Toronto • Singapore

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Library of Congress Cataloging in Publication Data:

White, R. E. (Robert E.)
An introduction to the finite element method with applications to nonlinear problems.

"A Wiley-Interscience publication."
Bibliography: p.
Includes index.
1. Finite element method. 2. Nonlinear theories.
I. Title.

TA347.F5W49 1985 620'.001'515353 85-6440
ISBN 0-471-80909-8

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

This text has evolved from lecture notes for my 1981–1985 spring semester courses on the finite element method. The students were mostly from the graduate-level engineering programs at North Carolina State University. Consequently, the most important objectives included (1) giving the student the ability to modify existing finite element codes or to create new codes, (2) giving the student some appreciation for the error estimates, and (3) giving a summary and illustration of nonlinear algorithms. The present text has been written so that readers can choose either the methods aspects or the theoretical considerations as their main interests.

At the end of every chapter I have indicated additional readings, and I have pointed to certain exercises that former students have found helpful. These include some programming problems as well as some theory problems. Readers will soon discover that these programming problems can be very time consuming; consequently, I recommend working with a partner. This helps the debugging process and gives students an opportunity to talk about the course.

The programs in this text are not meant to be optimal or elegant, but I hope they will be instructive. There are many optimized codes that one should try to use in "production" work. Any computer center should have relevant manuals, such as J. Rice's *Numerical Methods Software and Analysis: IMSL Reference Edition* (McGraw-Hill, 1983).

I would like to acknowledge the students, especially Maurizio Benassi, who have made many useful remarks on the contents of this book. Many thanks go to my friends who have listened to me concerning the more mundane aspects of writing this text. Finally, let me thank the staff of the mathematics department and, in particular, Nancy Burke, who did the typing of the manuscript.

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Raleigh, North Carolina
August 1985

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CONTENTS

INTRODUCTION

In this text we describe the finite element method with an emphasis on approximating solutions to second-order linear and nonlinear partial differential equations. The advantages of the finite element method over the finite difference method are (1) usually a more "accurate" approximation is obtained and (2) irregular shaped domains may be considered in the context of one program. The following examples illustrate the latter point.

Example 1. Ideal fluid flow around a pipe (see Figure 1). By the expected symmetry and the fact that the stream lines change most near the pipe, we may be interested in the nodes as distributed in

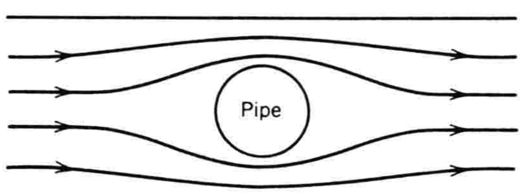
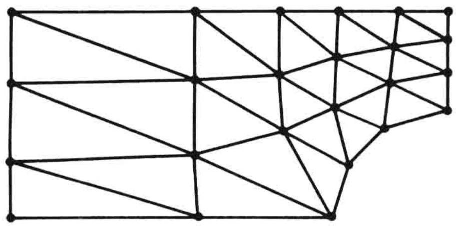


Figure 1



$\Delta\phi = 0$ on $\overline{\Omega}$
ϕ_y = velocity in x direction
$-\phi_x$ = velocity in y direction

Figure 2

Figure 2. Note $\overline{\Omega}$ is the union of the triangles and approximates Ω, the upper left region of fluid flow, more accurately than a union of rectangles with a similar number of nodes. The triangular regions are called elements for $\overline{\Omega}$.

Example 2. Steady-state heat flow. For example, consider an insulated steam pipe as illustrated in Figure 3. By using the symmetry we may reduce the number of nodes by $\frac{1}{8}$ (see Figure 4). The same

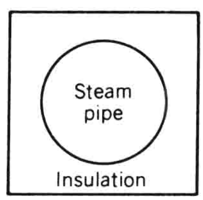


Figure 3

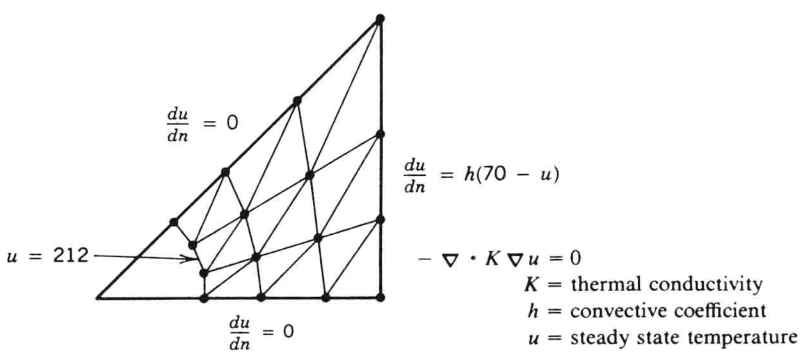


Figure 4

finite element program can be used to approximate the solution to both examples. One must input different data for both examples. Also, time-dependent problems may be considered.

Two main objectives of this text are (1) to present enough material so that readers can write their own finite element programs or alter existing codes and (2) to present some techniques for solving nonlinear problems. In the latter case we shall consider incompressible viscous fluid flow problems and nonlinear heat transfer problems such as the Stefan problem.

1

THE ENERGY AND WEAK FORMULATIONS

In order to introduce the finite element method (FEM), we consider a one-variable model problem. In the first three sections we illustrate the three equivalent formulations of this model problem. These are the classical, energy, and weak formulations. In Section 1.4 we discuss how they are related to one another. Sections 1.5 and 1.6 contain a description of two methods of assembling the system matrix, namely, assembly by nodes and assembly by elements. Section 1.7 contains a general outline for the finite element method.

1.1 THE CLASSICAL FORMULATION AND THE FINITE DIFFERENCE METHOD

The model problem that we shall use in this chapter is a mass subject to gravitational force and another force that is proportional to the displacement and whose positions at time $t = 0$ and $t = L$ are given. The classical formulation uses Newton's law and has the form

$$-m\ddot{y}(t) = mg - ky(t), \tag{1.1.1}$$

$$y(0) = a, \tag{1.1.2}$$

$$y(L) = b, \tag{1.1.3}$$

where m is the mass, k the proportionality constant, and g the acceleration due to gravity. $ky(t) - mg$ represents the external force.

There are other physical problems that have the same form as (1.1.1)–(1.1.3). For example, the steady-state deflection of an ideal string has the form

$$-(Tu_x(x))_x = f,$$

where T is the tension, u the displacement, and f the loading pressure. In this case the independent variable is a space variable x. Another example is steady-state heat conduction. A linearized version of the problem in exercise 1-25 has the form of (1.1.1).

Definition. We shall say that $y(t)$ is a *classical solution of the continuum problem* (1.1.1)–(1.1.3) if and only if $y \in C^2[0, L]$ and equations (1.1.1)–(1.1.3) are satisfied. ($C^2[0, L]$ is the set of functions on $[0, L]$ that have two continuous derivatives.)

Of course, if $m, k > 0$ are constants, then the classical solution of (1.1.1)–(1.1.3) is easy to find. If m, k are dependent on t, y or if equation (1.1.1) is more complicated, then one may not be able to find an explicit formula for the classical solution. One way to handle the more complicated problems is to approximate the continuum problem by a discrete model. The following is one such model called the finite difference method (FDM). Make the following approximations of y and $\ddot{y}$:

$$y(t) \to y_i \quad \text{where } y_i \simeq y(i\,\Delta t),\ \Delta t = L/N = h$$

and

$$\ddot{y}(t) \to \left(\frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h}\right)\frac{1}{h}.$$

Then equations (1.1.1)–(1.1.3) are approximated by

$$-m\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = mg - ky_i, \qquad 1 \le i \le N - 1 \qquad (1.1.4)$$

$$y_0 = a, \qquad (1.1.5)$$

$$y_N = b. \qquad (1.1.6)$$

For $m = k = 1$, $g = 32$, equation (1.1.4) may be written in the form

$$-\frac{1}{h}y_{i-1} + \left(\frac{2}{h} + h\right)y_i - \frac{1}{h}y_{i+1} = 32h. \tag{1.1.7}$$

Consequently, we have $N - 1$ unknowns and $N - 1$ equations. For $N = 4$, these may be written in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{-1}{h} & \frac{2}{h} + h & \frac{-1}{h} & 0 & 0 \\ 0 & \frac{-1}{h} & \frac{2}{h} + h & \frac{-1}{h} & 0 \\ 0 & 0 & \frac{-1}{h} & \frac{2}{h} + h & \frac{-1}{h} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} a \\ 32h \\ 32h \\ 32h \\ b \end{pmatrix}. \tag{1.1.8}$$

Definition. The discrete formulation of (1.1.1)–(1.1.3) given by (1.1.4)–(1.1.6) or in matrix form (for $N = 4$) by (1.1.8) is called the *finite difference model* of the classical formulation. The matrix in (1.1.8) is often called the *system matrix*.

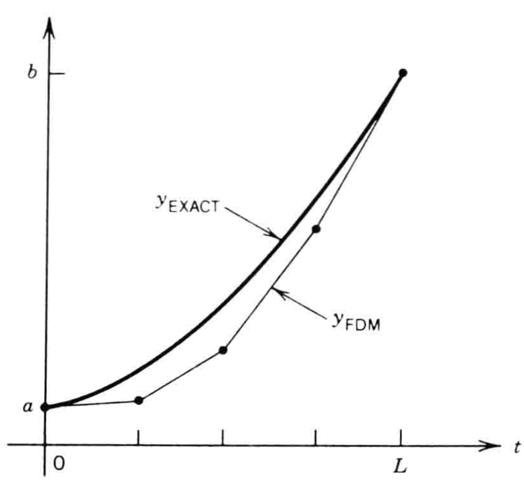


Figure 1.1.1