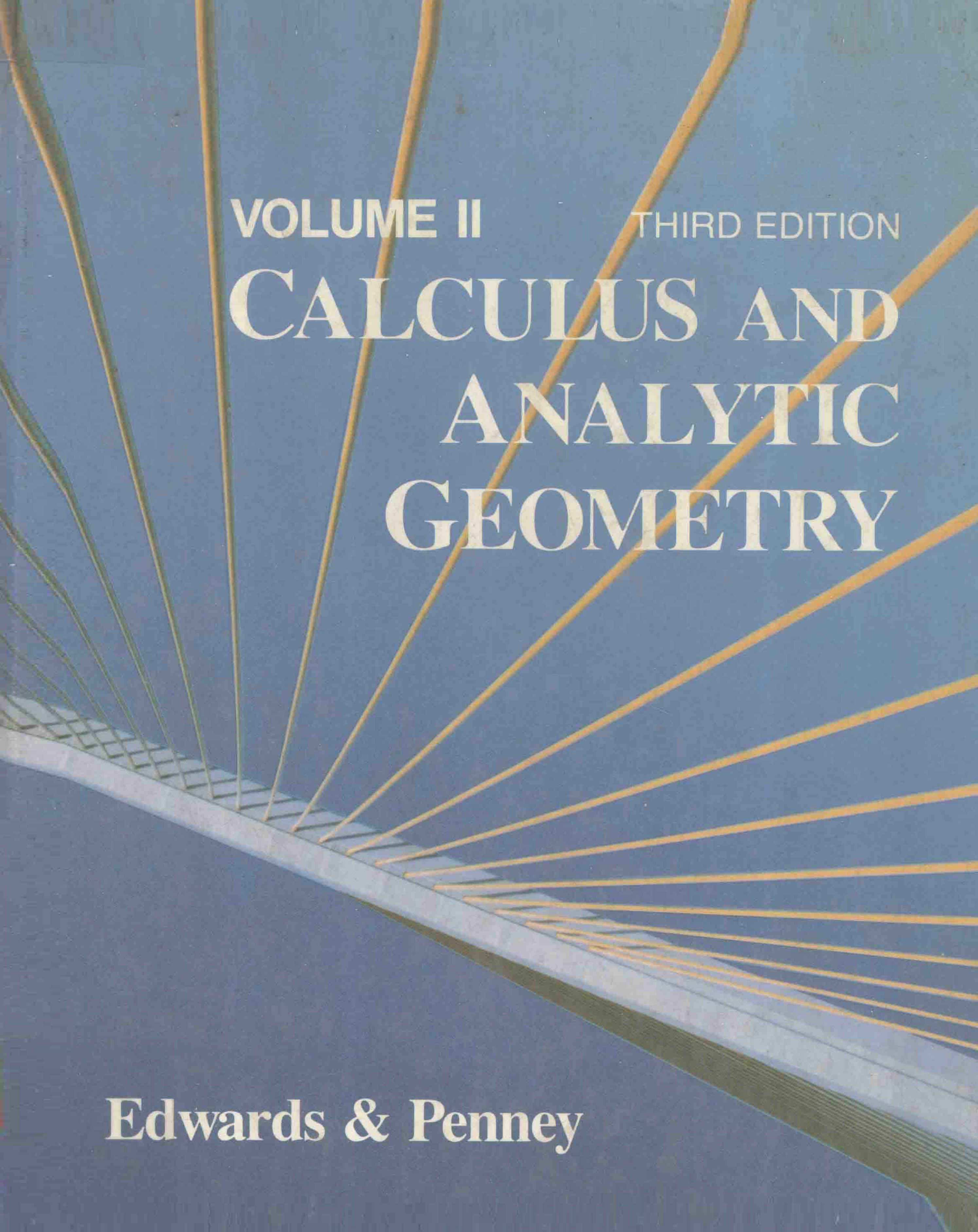

VOLUME II THIRD EDITION

CALCULUS AND ANALYTIC GEOMETRY

Edwards & Penney

Volume II

Calculus and Analytic Geometry

THIRD EDITION

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PRENTICE HALL
Englewood Cliffs, New Jersey, 07632

Library of Congress Catalog Card Number

90-60774

Design: Maureen Eide/Florence Silverman
Illustrator: Ron Weickart/Network Graphics
Manufacturing buyer: Paula Massenaro
Production: Nicholas Romanelli

Calculus and Analytic Geometry, Third Edition, Volume II
C. H. Edwards, Jr. and David E. Penney

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

ISBN 0-13-116583-6

Prentice-Hall International (UK) Limited, *London*
Prentice-Hall of Australia Pty. Limited, *Sydney*
Prentice-Hall of Canada, Inc., *Toronto*
Prentice-Hall Hispanoamericana, S.A., *Mexico*
Prentice-Hall of India Private Limited, *New Delhi*
Prentice-Hall of Japan, Inc., *Tokyo*
Simon & Schuster Asia Pte. Ltd., *Singapore*
Editora Prentice-Hall do Brasil, Ltda., *Rio de Janeiro*

Preface

For three centuries calculus has served as the prinicpal quantitative language of science and technology, and thereby has helped to shape the world in which we live. Today it remains a vibrant and living subject. Indeed, a wider range of students than ever before find a knowledge of the basic concepts of calculus necessary for their chosen courses of study.

Our goal in preparing this revision was to make the ideas of calculus more attractive and accessible to these many students for whom calculus is a keystone of their education. This edition is somewhat leaner and (we hope) crisper than its predecessors. Its text is about seventy-five pages shorter than that of the second edition, but we believe no one will find his or her favorite topic in the second edition missing from this one. Our main editorial technique has been the judicious pruning of excess foliage rather than the removal of whole trees.

This edition (like its predecessors) was written with five related objectives in constant view: *concreteness*, *readability*, *motivation*, *applicability*, and *accuracy*.

CONCRETENESS

The power of calculus is impressive in its precise answers to realistic questions and problems. In the necessary conceptual development of the subject, we keep in sight the central question: How does one actually *compute* it? We place special emphasis on concrete examples, applications, and problems that serve both to highlight the development of the theory and to demonstrate the remarkable versatility of calculus in the investigation of important scientific questions.

READABILITY

Difficulties in learning mathematics often are complicated by language difficulties. Our writing style stems from the belief that crisp exposition, both intuitive and precise, makes mathematics more accessible—and hence more readily learned—with no loss of rigor. We hope our language is clear and attractive to students and that they can and actually will read it, thereby enabling the instructor to concentrate class time on the less routine aspects of teaching calculus.

MOTIVATION

Our exposition is centered around examples of the use of calculus to solve real problems of interest to real people. In selecting such problems for examples and exercises, we took the view that stimulating interest and motivating effective study go hand in hand. We attempt to make it clear to students how the knowledge gained with each new concept or technique will be worth the effort expended. In theoretical discussions, especially, we try to provide an intuitive picture of the goal before we set off in pursuit of it.

APPLICATIONS

Its diverse applications are what attract many students to calculus, and realistic applications provide valuable motivation and reinforcement for all students. This book is well-known for the broad range of applications that we include, but it is neither necessary nor desirable that the course cover all the applications in the book. Each section or subsection that may be omitted without loss of continuity is marked with an asterisk. This provides flexibility for each instructor to determine his or her own flavor and emphasis.

ACCURACY

Our coverage of calculus is complete (though we hope it is somewhat less than encyclopedic). Like its predecessors, this edition was subjected to a comprehensive reviewing process to help ensure accuracy. With regard to the selection and sequence of mathematical topics, our approach is traditional. However, close examination of the treatment of standard topics will uncover evidence of our interest in the current movement to revitalize the teaching of calculus. We continue to favor an intuitive approach that emphasizes both conceptual understanding and care in the formulation of definitions and statements of theorems. Some proofs that may be omitted at the discretion of the instructor are placed at the ends of sections. Others (such as the proofs of the intermediate value theorem and of the integrability of continuous functions) are deferred to the book's appendices. In this way we leave ample room for variation in seeking the proper balance between rigor and intuition.

THIRD EDITION FEATURES

In preparing this edition, we have benefited from many valuable comments and suggestions from users of the first two editions. This revision was so pervasive that the individual changes are too numerous to be detailed in a preface, but the following paragraphs summarize those that may be of widest interest.

Additional Problems The number of problems has steadily increased since the first edition, and now totals over 6000. Most of the new problems are practice exercises that have been inserted near the beginnings of problem sets to insure that students gain sufficient confidence and computational skill before moving on to the more conceptual problems that constitute the real goal of calculus. However, we have continued to add occasional new conceptual and applied problems.

New Examples and Computational Details In many sections throughout this edition, we have inserted a simpler first example or have replaced existing examples with ones that are computationally simpler. Moreover, we have inserted an additional line or two of computational detail in many of the worked-out examples to make them easier for student readers to follow. The purpose of these computational changes is to make the computations themselves less of a barrier to conceptual understanding.

Optional Computer Applications We have included a dozen and a half optional computer notes for supplementary reading (or at least perusal) by students who might be motivated by computer applications. Each of these notes appears at the end of a section (following the problems) and uses some aspect of modern computational technology to illustrate the principal ideas of the section. These notes are completely optional and are never referred to in the text proper. Most of them apply very simple BASIC programming to calculus problems, but others range in content from illustrations of the use of handheld calculators with graphics capabilities to applications of symbolic algebra systems (such as MACSYMA, Maple, or Mathematica). The purpose of these notes is to stimulate interest in calculus in the rapidly increasing population of students who already are interested in computers. Those who would like to pursue computer applications further may consult the following books:

C. H. Edwards, Jr., *Calculus and the Personal Computer*, Englewood Cliffs, N. J.: Prentice Hall, 1986.

C. H. Edwards, Jr., *A Calculus Companion: The Personal Computer*, Englewood Cliffs, N. J.: Prentice Hall, 1990.

The latter is suitable for use in a computer lab that is conducted in association with a standard calculus course, perhaps meeting weekly. It can also be used as a basis for computer assignments that students will complete outside of class, for individual study or for supplementary projects.

Introductory Chapters Chapters 1 and 2 have been streamlined for a leaner and quicker start on calculus. Chapter 1 now consists of just three sections dealing with fundamental ideas about functions and graphs. Auxiliary ideas (e.g., inverse functions) now are deferred until later when they are actually needed. Chapter 2 on limits begins with a section on tangent lines to motivate the official introduction of limits in Section 2.2. The review of elementary trigonometry now appears as Appendix A (complete with an exercise set) at the back of the book, and can be used as a Chapter 2 insert where appropriate. The material on trigonometric limits is now delayed until it is needed in Chapter 3. Proofs of the limit laws are now included for reference in Appendix B.

Differentiation Chapters We have substantially reordered the sequence of topics in Chapters 3 and 4, with the objective of building student confidence by introducing topics more nearly in order of increasing difficulty. The chain rule now appears earlier (in Section 3.3) and we cover the basic techniques for differentiating algebraic functions before discussing maxima and minima in Sections 3.5 and 3.6. Implicit differentiation and related rates have been combined in a single section (Section 3.8). The mean value theorem and its applications are deferred to Chapter 4. Section 4.6 on higher derivatives and concavity has been streamlined, and applied problems using the second derivative test have been added.

Integration Chapters New and simpler examples, together with enhanced artwork, have been inserted throughout Chapters 5 and 6. Many instructors now believe that first applications of integration ought not be confined to the standard area and volume computations; Section 6.5 is an optional new section that introduces separable differential equations. The material on centroids and the theorems of Pappus has been moved to Chapter 16 (Multiple Integrals) where it can be treated in a more natural context.

Transcendental Functions Chapter 7 now begins with a more intuitive introduction to exponential and logarithmic functions that is based on the precalculus idea of a logarithm as "the power to which the base *a* must be raised to get the number *x*." On this basis, Section 7.1 carries out a low-key review of the laws of exponents and of logarithms, and investigates informally the differentiation of exponential and logarithmic functions. This informal discussion, together with much-needed review of precalculus material, is intended to provide students with a conceptual foundation for the "official" treatments of these functions that appear in Sections 7.2 and 7.3. Many of the lengthier applications that formerly appeared in Chapters 7 and 8 have been deleted, but sufficiently many remain to amply illustrate the role of transcendental functions in the real world.

Techniques of Integration The first four sections (through integration by parts) of Chapter 9 are essentially unchanged, but the remainder of the chapter has been reorganized for the benefit of those instructors who feel that methods of formal integration now require less emphasis, in view of modern techniques for both numerical and symbolic integration. The method of partial fractions now appears in Section 9.5 (immediately following integration by parts in Section 9.4). Trigonometric substitutions and integrals involving quadratic polynomials follow in Sections 9.6 and 9.7. This rearrangement of Chapter 9 makes it more convenient to stop wherever the instructor desires.

Multivariable Topics Section 14.5 is a unified treatment of curvature and acceleration for both plane curves and space curves (which were treated separately in the second edition). The discussion of directional derivatives in Section 15.8 has been simplified considerably. Section 16.5 (Applications of Double Integrals) now contains the theorems of Pappus that formerly were discussed in Chapter 6.

Differential Equations Many calculus instructors now believe that differential equations should be seen as early and as often as possible. The very simplest differential equations (of the form $y' = f(x)$) appear in an optional subsection at the end of Section 4.8 (Antiderivatives). Section 6.5 is a new section that illustrates applications of integration to the solution of separable differential equations. However, these are optional sections, and the instructor can delay them until Chapter 18 (on differential equations) is covered. Chapter 18 has been revised substantially, and now ends with Section 18.6 on elementary numerical methods. Some of the lengthier applications in earlier editions have been deleted but are now included in Edwards and

Penney, *Elementary Differential Applications with Applications*, second edition (Englewood Cliffs, N.J.: Prentice Hall, 1989).

Computer Graphics The ability of students to visualize surfaces and graphs of functions of two variables should be enhanced by the computer graphics that appear in Chapters 14 through 16. All of these computer-generated figures either are new or were reworked for the third edition. We are indebted to John K. Edwards for the "hard copy" computer graphics and to Roy Myers for the photographs of monitor screens generated by his Calculus 3–D Function Plotter.

Historical Comments Both authors are fond of the history of mathematics, and believe that it can favorably influence our teaching of mathematics. For this reason numerous historical comments appear in the text. However, we have resisted the temptation to insert full-fledged historical notes. Instructors who would like to include more historical material in their courses are invited to consult Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979).

SUPPLEMENTARY MATERIAL

Answers to most of the odd-numbered problems appear in the back of the book. Solutions to most problems (other than those odd-numbered ones for which an answer alone is sufficient) are available in the *Instructor's Manual*. A subset of this manual, containing solutions to problems numbered 1, 5, 9, 13, . . . is available as a *Student Manual*. A collection of some 1400 additional problems suitable for use as test questions, the *Calculus Test Item File*, is available for use by instructors. Finally, an *Instructor's Annotated Edition* of the text itself is available to those who are teaching from this book. A computer diskette that accompanies the instructor's edition includes memos of possible interest to instructors as well as the BASIC programs that appear in the optional computer notes in the text itself. A variety of additional supplements are provided by the publisher.

Acknowledgments

All experienced textbook authors know the value of critical reviewing during the preparation and revision of a manuscript. In our work on various editions of this book we have benefited greatly from the advice (and frequently the consent) of the following exceptionally able reviewers:

Leon E. Arnold, Delaware County Community College
H.L. Bentley, University of Toledo
Michael L. Berry, West Virginia Wesleyan College
William Blair, Northern Illinois University
George Cain, Georgia Institute of Technology
Wil Clarke, Atlantic Union College
Peter Colwell, Iowa State University
James W. Daniel, University of Texas at Austin
Robert Devaney, Boston University
Dianne H. Haber, Westfield State College
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Catherine Lilly, Westfield State College
Barbara Moses, Bowling Green University
Barbara L. Osofsky, Rutgers University—New Brunswick
James P. Qualey, Jr., University of Colorado
Thomas Roe, South Dakota State University
Lawrence Runyan, Shoreline Community College
John Spellman, Southwest Texas State University
Virginia Taylor, University of Lowell
Samuel A. Truitt, Jr., Middle Tennessee State University
Robert Urbanski, Middlesex County College
Robert Whiting, Villanova University

Many of the best improvements that have been made must be credited to colleagues and users of the first two editions throughout the country (and abroad). We are grateful to all those, especially students, who have written to us, and hope they will continue to do so. We also believe that the quality of the finished book itself is adequate testimony to the skill, diligence, and talent of an exceptional staff at Prentice-Hall; we owe special thanks to Bob Sickles, mathematics editor; Nick Romanelli, production editor; Maureen Eide and Florence Silverman, designers; and Ron Weickart, illustrator. Finally, we again are unable to thank Alice Fitzgerald Edwards and Carol Wilson Penney adequately for their continued assistance, encouragement, support, and patience.

Athens, Georgia C.H.E., Jr. / D.E.P.

Contents

Volume I consists of Chapters 1–12 and all the front and backmatter of the original one-volume Calculus and Analytic Geometry, *3rd Ed. Volume II consists of Chapters 12–18 and all the front and backmatter.*

chapter twelve

Infinite Series

12.1 Introduction

In the fifth century B.C., the Greek philosopher Zeno proposed the following paradox: In order for a runner to travel a given distance, the runner must first travel halfway, then half the remaining distance, then half the distance that yet remains, and so on ad infinitum. But, Zeno argued, it is clearly impossible for a runner to accomplish these infinitely many steps in a finite period of time, so motion from one point to another must be impossible.

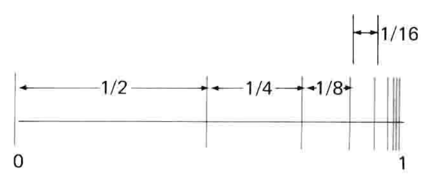
Zeno's paradox suggests the infinite subdivision of [0, 1] indicated in Fig. 12.1. There is one subinterval of length $1/2^n$ for each integer $n = 1, 2, 3, \ldots$. If the length of the interval is the sum of the lengths of the subintervals into which it is divided, then it would appear that

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots,$$

with infinitely many terms somehow adding up to 1. On the other hand, the formal infinite sum

$$1 + 2 + 3 + \cdots + n + \cdots$$

of all the positive integers seems meaningless—it does not appear to add up to *any* (finite) value.



12.1 Subdivision of an interval to illustrate Zeno's paradox

The question is this: What, if anything, is meant by the sum of an *infinite* collection of numbers? This chapter explores conditions under which an *infinite* sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

known as an *infinite series*, is meaningful. We shall discuss methods for computing the sum of an infinite series, and applications of the algebra and calculus of infinite series. Infinite series are important in science and mathematics because many functions either arise most naturally in the form of infinite series, or have infinite series representations (such as the Taylor series of Section 12.7) that are useful for numerical computations.

12.2 Infinite Sequences

An **(infinite) sequence** of real numbers is a function whose domain of definition is the set of all positive integers. Thus if s is a sequence, then to each positive integer n there corresponds a real number $s(n)$. Ordinarily, a sequence is most conveniently described by listing its values in order, beginning with $s(1)$:

$$s(1),\ s(2),\ s(3),\ \ldots,\ s(n),\ \ldots.$$

With subscript notation rather than function notation, we may write

$$s_1,\ s_2,\ s_3,\ \ldots,\ s_n,\ \ldots \tag{1}$$

for this list of values. The values in this list are the **terms** of the sequence; s_1 is the first term, s_2 the second term, s_n the ***n*th term.**

We use the notation $\{s_n\}_{n=1}^{\infty}$, or simply $\{s_n\}$, as an abbreviation for the **ordered** list in (1), and we may refer to the sequence by saying simply "the sequence $\{s_n\}$." When a particular sequence is so described, the nth term s_n is generally (though not always) given by a formula in terms of its subscript n. In this case, listing the first few terms of the sequence often helps us to see it more concretely.

EXAMPLE 1 The following table lists explicitly the first four terms of each of several sequences.

$\{s_n\}_1^{\infty}$	$s_1, s_2, s_3, \ldots$
$\left\{\frac{1}{n}\right\}_1^{\infty}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
$\left\{\frac{1}{10^n}\right\}_1^{\infty}$	$0.1, 0.01, 0.001, 0.0001, \ldots$
$\left\{\frac{1}{n!}\right\}_1^{\infty}$	$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots$
$\left\{\sin\frac{n\pi}{2}\right\}_1^{\infty}$	$1, 0, -1, 0, \ldots$
$\{1+(-1)^n\}_1^{\infty}$	$0, 2, 0, 2, \ldots$

EXAMPLE 2 The *Fibonacci sequence* $\{F_n\}$ is defined as follows:

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for} \quad n \geqq 2.$$

The first ten terms of the Fibonacci sequence are

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55.$$

This is a *recursively defined* sequence—after the initial values are given, each term is defined in terms of its predecessors.

The limit of a sequence is defined in much the same way as the limit of an ordinary function (Section 2.2).

Definition *Limit of a Sequence*
We say that the sequence $\{s_n\}$ **converges** to the real number L, or has **limit** L, and we write

$$\lim_{n\to\infty} s_n = L, \tag{2}$$

provided that s_n can be made as close to L as we please by choosing n sufficiently large. That is, given any number $\varepsilon > 0$, there exists an integer N such that

$$|s_n - L| < \varepsilon \quad \text{for all} \quad n \geqq N. \tag{3}$$

EXAMPLE 3 Prove that $\lim_{n\to\infty} \dfrac{1}{n} = 0.$

Proof We need to show this: To each positive number ε, there corresponds an integer N such that, for all $n \geqq N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

It suffices to choose any fixed integer $N > 1/\varepsilon$. For example, merely let $N = 1 + [\![1/\varepsilon]\!]$. Then $n \geqq N$ implies that

$$\frac{1}{n} \leqq \frac{1}{N} < \varepsilon,$$

as desired.

The limit laws stated in Section 2.2 for limits of functions have natural analogues for limits of sequences. Their proofs are based on techniques similar to those used in Appendix B.

Theorem 1 *Limit Laws for Sequences*

If the limits

$$\lim_{n\to\infty} a_n = A \quad \text{and} \quad \lim_{n\to\infty} b_n = B$$

exist (so that A and B are real numbers), then:

1. $\lim_{n\to\infty} ca_n = cA$ (c any real number);
2. $\lim_{n\to\infty} (a_n + b_n) = A + B$;
3. $\lim_{n\to\infty} a_n b_n = AB$;
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.

In this last case we must assume also that $B \neq 0$ and that $b_n \neq 0$ for all sufficiently large values of n.

Theorem 2 *Substitution Law for Sequences*

If $\lim_{n\to\infty} a_n = A$ and the function f is continuous at $x = A$, then

$$\lim_{n\to\infty} f(a_n) = f(A).$$

Theorem 3 *Squeeze Law for Sequences*

If $a_n \leqq b_n \leqq c_n$ for all n and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\lim_{n\to\infty} b_n = L$ as well.

These theorems can be used to compute limits of many sequences formally, without recourse to the definition. For example, if k is a positive integer and c is a constant, then Example 3 and the product law (Theorem 1, part 3) give

$$\lim_{n\to\infty} \frac{c}{n^k} = c \cdot 0 \cdot 0 \cdots 0 = 0.$$

EXAMPLE 4 Show that $\lim_{n\to\infty} \frac{(-1)^n \cos n}{n^2} = 0$.

Solution This result follows from the squeeze law and the fact that $1/n^2 \to 0$ as $n \to \infty$, because

$$-\frac{1}{n^2} \leqq \frac{(-1)^n \cos n}{n^2} \leqq \frac{1}{n^2}.$$

EXAMPLE 5 Show that if $a > 0$, then $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Solution We apply the substitution law with $f(x) = a^x$ and $A = 0$. Because $1/n \to 0$ as $n \to \infty$ and f is continuous at $x = 0$, this gives

$$\lim_{n\to\infty} a^{1/n} = a^0 = 1.$$

EXAMPLE 6 The limit laws and the continuity of $f(x) = \sqrt{x}$ at $x = 4$ yield

$$\lim_{n\to\infty} \sqrt{\frac{4n-1}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{4-(1/n)}{1+(1/n)}} = \sqrt{4} = 2.$$

EXAMPLE 7 Show that if $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$.

Solution Because $|r^n| = |(-r)^n|$, we may assume that $0 < r < 1$. Then $1/r = 1 + a$ with $a > 0$, so the binomial formula yields

$$\frac{1}{r^n} = (1+a)^n = 1 + na + \{\text{positive terms}\} > 1 + na,$$

so

$$0 < r^n < \frac{1}{1+na}.$$

Now $1/(1+na) \to 0$ as $n \to \infty$. Therefore, the squeeze law implies that $r^n \to 0$ as $n \to \infty$.

Let f be a function defined for every real number $x \geqq 1$, and $\{a_n\}$ a sequence such that $f(n) = a_n$ for every positive integer n. Then it follows from the definitions of limits of functions and sequences that

$$\textit{if} \quad \lim_{x\to\infty} f(x) = L, \quad \textit{then} \quad \lim_{n\to\infty} a_n = L. \tag{4}$$

Note that the converse of the statement in (4) is generally false. For example,

$$\lim_{n\to\infty} \sin \pi n = 0 \quad \text{but} \quad \lim_{x\to\infty} \sin \pi x \text{ does not exist.}$$

Because of (4) we can use **l'Hôpital's rule for sequences:** If $a_n = f(n)$, $b_n = g(n)$, and $f(x)/g(x)$ has the indeterminate form ∞/∞ as $x \to \infty$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}, \tag{5}$$

provided that f and g satisfy the other hypotheses of l'Hôpital's rule, including the assumption that the right-hand limit exists.

EXAMPLE 8 Show that $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

Solution The function $(\ln x)/x$ is defined for all $x \geqq 1$ and agrees with the given sequence when $x = n$, a positive integer. Because $(\ln x)/x$ has the indeterminate form ∞/∞ as $x \to \infty$, l'Hôpital's rule gives

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$