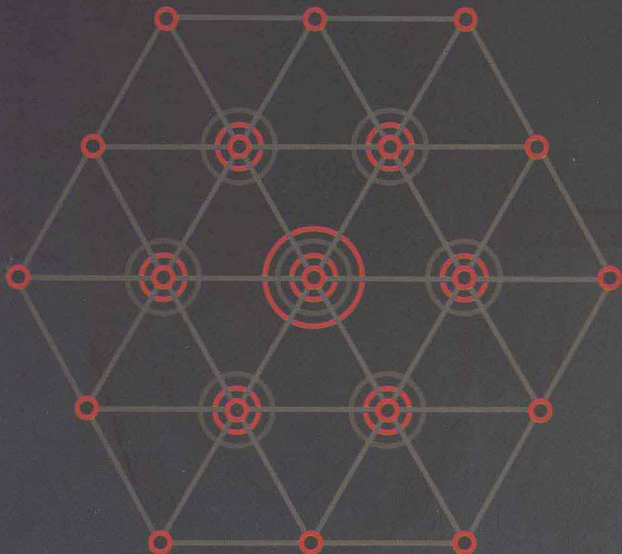

Algebraic Groups and Homogeneous Spaces

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Homogeneous Spaces
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International Colloquium
on
Algebraic Groups and Homogeneous Spaces
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Report

An International Colloquium on 'Algebraic Groups and Homogeneous Spaces' was held at the Tata Institute of Fundamental Research, Mumbai from January 6 to January 14, 2004. This was one of the quadrennial colloquia organised by the School of Mathematics, TIFR. This colloquium was co-sponsored by the International Mathematical Union and the Indian National Science Academy, and financial support came from the IMU and the Sir Dorabji Tata Trust. The Colloquium was a J.R.D. Tata Birth Centenary Year event.

The Organising Committee consisted of Professors S. M. Bhatwadekar (Chair), I. Biswas, P.A. Griffiths (IMU Nominee), V.B. Mehta, Arvind Nair, A.J. Parameswaran, M.S. Raghunathan (IMU Nominee) and C.S. Rajan.

There were 29 one-hour talks by invited speakers on topics related to the Colloquium. Besides the invited speakers, the members of the School of Mathematics and a few mathematicians from Universities and educational institutions in India and abroad attended the Colloquium.

On Wednesday, 7 January 2004, there were two talks in memory of Armand Borel. T.A. Springer gave a talk on Armand Borel's work on algebraic groups and C.S. Seshadri gave a talk on Armand Borel and Indian Mathematics. Professor Borel had accepted the Organizing Committee's invitation to speak at the Colloquium but unfortunately passed away in August 2003.

The social programme for the Colloquium included a Tea on the East Lawns of the Institute on 6 January, 2004, a dinner party on 10 January, 2004 and an excursion to Karjat on 11 January, 2004. The cultural events included a Bharata Natyam recital on 12 January, 2004 and a Hindustani vocal recital on 8 January, 2004.

List of Talks

Speaker	Title
H. H. Andersen	Cohomology of line bundles
V. Balaji	Geometry of principal bundles
P. Belkale	The Horn conjecture and representations of the fundamental group
I. Biswas	Higgs fields and flat connections on a principal bundle over a compact Kähler manifold
M. Brion	Normality of certain orbit closures in group completions
V. Chari	Representations of quantum affine algebras
J.-L. Colliot-Thélène	Zero-dimensional Chow group of linear algebraic groups over local fields
P. Deligne	Tensor categories
H. Garland	Eisenstein series on certain infinite-dimensional, arithmetic quo tients
Y. I. Holla	Extension of structure groups of principal bundles in positive characteristic
V. Kac	Quantum Hamiltonian reduction for affine (super)algebras
W. van der Kallen	Extending the finite generation theorem of invariant theory to cohomology
S. Kannan	Cohomology of line bundles on Schubert varieties
F. Knop	Affine Hamiltonian varieties
Shrawan Kumar	Induction functor in non-commutative equivariant cohomology and Dirac cohomology
V. Lakshmibai	A standard monomial basis for the variety of nilpotent matrices
E. Looijenga	Ball quotients originating from finite complex reflection groups

Speaker	Title
O. Mathieu	On modular unipotent representations
T. Miwa	Physical combinatorics
Arvind Nair	Lefschetz theorems and arithmetic quotients
H. Nakajima	Crystal bases of quantum affine algebras
R. Parimala	Homogeneous spaces under linear algebraic groups over fields of cohomological dimension two
R. Parthasarathy	Quantum analogues of coherent modules – A positivity result for G_2
V. L. Popov	Cayley maps for algebraic groups
C. Procesi	Cayley–Hamilton algebras and representation theory
C. S. Rajan	Unique decomposition of tensor products of irreducible representations of simple algebraic groups
Arun Ram	The role of affine Hecke algebras in the combinatorial representation theory of reductive algebraic groups
T. A. Springer	Some subvarieties of the compactifications of a semi-simple group
S. Subramanian	Principal bundles on the projective line

Contents

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Armand Borel's Work in the Theory of Linear Algebraic Groups

T.A. Springer

The following text is based on a talk which I gave in the Colloquium. It deals with the work of Armand Borel on the basic theory of linear algebraic groups. I will not go into his important work on the applications of the theory to arithmetic groups and automorphic forms.

I have borrowed from the memorial article in the Notices of the American Mathematical Society, vol. 51 (2004), p. 498–524, which gives a more complete picture of Borel and his work.

1 Biographical Facts

Armand Borel, was born in La Chaux-de-Fonds (Switzerland) in 1923. He studied mathematics at the Eidgenössische Technische Hochschule (ETH) in Zürich from 1942-1947. He continued his studies in Paris, in 1952 he obtained the doctorate at the University of Paris. Soon afterwards he went to the United States (Princeton 1952–54, Chicago 1955–56). Since 1957 he was professor at the Institute for Advanced Study in Princeton N.J. (emeritus since 1993). He died in Princeton in August of 2003, after a brief illness.

2 Early Work

At the ETH Borel had distinguished teachers: H. Hopf in topology and E. Stiefel in the theory of Lie groups. Later, in Paris Borel became one of the first to fathom the theory of "spectral sequences", introduced by J. Leray in algebraic topology. In his thesis [B, 23]* Borel applies spectral sequences, to the study of the cohomology with integer coefficients of a homogeneous space K/H of a (connected) compact Lie group K.

*This refers to the numbering of the papers in [B, IV]. The cited papers are listed in the References.

In Princeton in 1952–54 he started a collaboration with F. Hirzebruch on the algebraic topology of the space K/H, in particular on their characteristic classes (published in [B, 43, 45, 47]). In the case of a maximal torus H application of Hirzebruch's Riemann-Roch theorem to line bundles on the complex projective algebraic variety K/H led to an appearance of H. Weyl's formula for the dimension of an irreducible representation of K. At this point it was a short step to the Borel-Weil theorem according to which the irreducible representations of K are realized in spaces of sections of line bundles on K/H. The theorem was obtained by Borel and Weil in 1953, but was not published at that time (Borel's note [B, 30] on the subject appeared only in 1983).

3 Linear Algebraic Groups

Borel started the work on his fundamental paper on the theory of linear algebraic groups ("Groupes linéaires algébriques" [B, 39]) during his stay at the University of Chicago (1955-56). I don't know what was the incentive for Borel to take up the subject, which was a new line of research for him. The subject was in the air: Kolchin (1948) and Chevalley (1951) had already made some basic contributions. Also, Borel might have felt the need for an extension of the theory of Lie groups in which arithmetic questions could be dealt with. [B, 22] shows that he had an early interest in the theory of automorphic functions, where such arithmetic questions appear naturally.

In [B, 39] Borel develops the theory with characteristic thoroughness, using global algebro-geometric methods (in the context of algebraic geometry à la A. Weil). An important ingredient for him was Weil's work (published around the same time) on the construction of homogeneous spaces of algebraic groups, i.e., the construction of the quotient of an algebraic group by an algebraic subgroup.

Perhaps Borel was also influenced by the work of his teacher Heinz Hopf, who had introduced global geometric methods in the theory of compact Lie groups, to circumvent the use of Lie algebras.

Let k be an algebraically closed field, of arbitrary characteristic. The definition of a linear algebraic group over k given in [B, 39] is the usual one (more or less, as I have suppressed the use of a "universal domain"): a Zariski-closed subgroup of some $GL_n(k)$. Let G be an algebraic group over k[†]. I shall briefly review the main results of [loc. cit.].

Borel first establishes a number of elementary results which I will not go into.

[†] I drop the adjective "linear"

Tori are introduced in §7. In the introduction of the paper it is pointed out that their role is similar to the role played by the usual tori in the theory of compact Lie groups.

In §8 the Jordan decomposition $g = g_s g_u$ of an element $g \in G$ into commuting semisimple and unipotent elements is established.

Chapter III is devoted to the study of solvable groups. Let G be connected and solvable. The main results about such groups are:

(10.1) The set G_u of unipotent elements of G is a connected, normal, algebraic subgroup;
(12.2) Two maximal tori of G are conjugate and if T is one of them, G is the semi-direct product of T and the normal subgroup G_u;
(13.2) The centraliser of a subtorus of G is connected.

The proofs of these results are essentially elementary and were not surprising.

4 Borel Groups

The surprising new results of the paper were those about Borel groups, introduced (as yet unbaptized) in Chapter IV. Assume G to be connected (for simplicity). A Borel group B of G is, as everybody knows nowadays, a closed solvable algebraic subgroup of G, which is maximal relative to these properties. Borel establishes their properties:

(16.5) G/B is a projective algebraic variety;
(16.5) any two Borel subgroups of G are conjugate;
(17.4) the Borel groups cover G.

A main ingredient in the proof of the first two results is the "orbit lemma": if the algebraic group G acts on the algebraic variety X (in the sense of algebraic geometry) there exists a point of X whose G-orbit is closed. Nowadays this is viewed as an easy elementary fact. But in the 1950's it was a surprising fact: people were more familiar with complex analytic geometry and there such a fact is completely false.

The orbit lemma is also used to prove "Borel's fixed point theorem": a connected solvable linear algebraic group acting on a complete variety fixes a point of X. The theorem implies the conjugacy of Borel groups.

The proof of the third property uses the following construction (Borel formulates things a bit differently). Let $\tilde{G}$ be the quotient of $G \times B$ by the B-action $b.(g, b') = (gb^{-1}, bb'b^{-1})$ and let $\pi : \tilde{G} \to G$ be the morphism induced by the map $(g, b) \mapsto gbg^{-1}$. The image of π is the union of all

Borel groups. So the property asserts that π is surjective. As π is a proper morphism, surjectivity will follow if the image of π is dense. Borel shows this by proving that the union of the conjugates of a suitably chosen subgroup of B is dense.

The map π appears for the first time, implicitly, in Borel's work. Further study of π and of its fibers has led to interesting insights, discussed in [Slo].

Applications of the above properties are:

(16.6) any two maximal subtori of G are conjugate;

(18.1) a semi-simple element of G lies in a maximal torus;

(19.1) the centralizer of a subtorus of G is connected.

The proofs are reduced to the case of a solvable group by suitable use of (17.4).

(16.6) is an analog of the conjugacy of Cartan sungroups of a compact Lie group. Borel clearly was aware of the importance of, seemingly innocuous, connectedness results such as (19.1).

The last Chapter V of the paper deals with Cartan subgroups, which I will not go into.

What is not proved in the paper and what Borel did not know at the time of writing is the normalizer theorem: a Borel group of G coincides with its normalizer.

5 Chevalley's Work

Chevalley proved the normalizer theorem a little later and then developed a structure theory of semisimple groups. He gave a complete classification of simple algebraic groups over any algebraically closed field.

Borel tells in [B4, p. 158] that he gave Chevalley a copy of his paper in the summer of 1955. The next summer Chevalley told him that after reading the paper he had proved the normalizer theorem, after which the rest followed "by analytic continuation".

Chevalley also introduced the combinatorial ingredients from Lie theory, such as root system and Weyl group. His work was published in the Paris Seminar Notes [Che]; they were for many years the standard text for the theory of algebraic groups.

The Notes also contain (without naming it) the first published version of the Borel-Weil theorem in the context of algebraic groups [Che, exp. 15], which asserts that in characteristic zero the irreducible representations of a semisimple algebraic group G can be realized in spaces of sections of line bundles on G/B, where B is a Borel subgroup. Borel and Weil's analytic version of the theorem, involving compact Lie groups, was mentioned before. In the meantime it has become clear that the representation theory of compact Lie groups is equivalent to the representation theory of reductive algebraic groups over $\mathbb{C}$.

6 Ground Fields

The reviewed results by Borel and Chevalley deal with an algebraic group G over an algebraically closed field k. Let F be a subfield of k. The results can be refined by assuming that G and the morphisms defining the group structure are defined over F, in the sense of algebraic geometry. I shall then say that G is an F-group. Its group of F-rational points is denoted by $G(F)$. [B, 39] already contains several general facts on F-groups.

An interesting case is $F = \mathbb{Q}$, $k = \mathbb{C}$. Then $G(\mathbb{R})$ is a Lie group with subgroup $G(\mathbb{Q})$, via which arithmetic can be introduced on Lie groups. Here is the origin of Borel's work on the applications of algebraic group theory to arithmetic groups and automorphic forms, which I don't discuss. I only mention one of the first applications: the fundamental paper [B, 58] of 1962 by Borel and Harish-Chandra on reduction theory.

At about the same time Borel started a thorough study of properties of F-groups involving the ground field F. He expounded his work in a seminar at the Institute for Advanced Study in 1961-62. He had to restrict himself to the case of a perfect field F. This is an undesirable restriction, as it excludes interesting fields such as global fields of nonzero characteristic.

A little later the restriction could be removed thanks to work of Grothendieck to which I shall return below.

Most of Borel's later papers on algebraic groups were written in collaboration. This is not surprising: he had contacts with many mathematicians in the field, he gave them good advice and he shared his ideas.

7 Collaboration With Tits

In his work of F-groups Borel joined forces with J. Tits who had come to the study of F-groups from another point of view. The result of their collaboration was the fundamental paper "Groupes réductifs" [B, 66]. It contains a wealth of interesting material, I restrict myself to part of it.

Recall that a parabolic subgroup P of the algebraic group G is an algebraic subgroup containing a Borel subgroup or equivalently, such that G/P is a complete variety.

Now let G be reductive. In the theory over F the role of Borel subgroups over k is taken over by the minimal parabolic F-subgroups. It is shown that two of these are conjugate by an element of $G(F)$. If there are no proper parabolic F-subgroups the F-group G is called anisotropic. For example, such is the orthogonal group defined by an anisotropic quadratic form over F (whence the name).

The role of maximal tori over k is taken over by subtori of G which are defined over F and F-split, i.e. F-isomorphic to a group of diagonal matrices, and maximal for these properties. Let S be such a torus. Then G is anisotropic if and only if S lies in the center of G. This implies that the centralizer M of S, which is a connected reductive F-group, is anisotropic. Out of M and a half-space in the character group of S - which is a free abelian group of rank $\dim S$- one can construct a minimal parabolic subgroup over F. The split torus S defines a "small" Weyl group, an ingredient of a Tits system on $G(F)$.

It is also shown that two maximal split F-tori are $G(F)$-conjugate. In fact, this is true for any F-group G, not necessarily reductive. The proof is sketched in the later paper [B, 110].

8 Grothendieck's Work

I come now to the work of Grothendieck, alluded to above, which enabled Borel and Tits to establish their results for an arbitrary base field F. Grothendieck's work (from 1964) was expounded in his Seminar on Group Schemes, see [SGA3, exp. XII, XIII, XIV]). New results were:

(a) an F-group G contains a maximal torus which is defined over F,

(b) if moreover G is connected, reductive and F is infinite then $G(F)$ is Zariski-dense in G.

His proofs, using scheme-theoretic techniques, were outside the realm of usual algebraic geometry, which was unsatisfactory to Borel. In work with T. A. Springer [B, 76, 80] he managed to give proofs along more traditional lines.

By the way, Grothendieck did make use in his proofs of a traditional object, namely the Lie algebra of G, whose use in the theory of algebraic groups over fields of nonzero characteristic had so far been avoided.

9 Further Work With Tits

Borel continued his collaboration with Tits. In [B, 92] it is shown how to associate canonically to a unipotent subgroup U of a reductive group G a parabolic subgroup whose unipotent radical contains U (in [loc. cit.] fields of definition are also taken into account). A consequence is the following nice result (which was known in characteristic 0): a maximal proper closed subgroup of G is either parabolic or reductive.

The starting point of the long joint paper [B, 97] is the problem of determining the automorphisms of the abstract group $G(F)$, where G is a connected semisimple F-group. More generally, one can study the homomorphisms of $G(F)$ into a similar group $G'(F')$. The problem had been around since the 1920's and had been solved in many particular cases, under restrictions on G or F (see [B1, p. 134-140]).

G is assumed to be semisimple. The important standing hypothesis is: G is isotropic over F, so contains proper parabolic F-subgroups. Let G^+ be the subgroup of $G(F)$ generated by the groups $U(F)$, where U runs through the unipotent radicals of the parabolic F-subgroups of G (G^+ is a "large" subgroup of $G(F)$ but does not always coincide with it). If $\phi : F \to F'$ is a homomorphism of fields one can transport G via ϕ to a group ${}^{\phi}G$ over F' and there is a canonical homomorphism $\phi^0 : G(F) \to {}^{\phi}G(F')$.

I shall not try to fully describe the results of [B, 97]. Here is a typical example. Assume that G is simple (and isotropic over F). Let G' be a simple (non-trivial) algebraic group over F' and let $\alpha : G(F) \to G'(F')$ be a homomorphism such that $\alpha(G^+)$ is Zariski-dense in G'. Then there is a homomorphism $\phi : F \to F'$ and an isomorphism of F'-groups $\beta : {}^{\phi}G \to G'$ such that $\alpha(g) = \beta(\phi^0(g))$ for $g \in G^+$.

Another topic of the paper is the analysis of an irreducible representation (in the algebraic sense) $\rho : G(F) \to PGL_n(k')$, k' being algebraically closed. It is shown that ρ can be built up from irreducible projective representations of the algebraic group G.

The paper exploits the properties of parabolic subgroups established in [B, 66]. There are many technicalities, in particular in characteristic 2.

The restriction to isotropic groups made in [B, 97] is very essential. For the case of anisotropic groups there is, as far as I know, as yet no general theory.

The short note [B, 110] of Borel and Tits announces a number of results about a connected algebraic group G over a non-perfect field F which is "reductive relative to F", i.e. which has no nontrivial connected normal unipotent closed F-subgroup (the assumptions in [loc. cit.] are a bit more general). Analogues of the results of [B, 66] are announced, such as a theory of "pseudo-parabolic" subgroups and the existence of a Tits system on $G(F)$. Subsequently, Tits has continued the work, leading to classification results. He has lectured about these matters at the Collège de France. But full proof of his results and of those of [B, 110] have not appeared. (Some of the results of Borel and Tits are treated in [Sp, Ch. 15].)