ROBERT T. SEELEY

Calculus of One Variable

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Editor's Preface

Most calculus texts can be put into one of two classes. In the first class belong those books whose main concerns are the formal rules for differentiation and integration and their many applications to physics and geometry. By and large, these texts ignore the foundations of the subject and try to hide the intellectual and technical subtleties of the limit process, the real numbers, order of growth, and inequalities. The successful student emerges with a good technique for solving the standard problems that have made the calculus such an important subject. However, subtler applications are beyond him because the foundations have not been exposed to him.

In the second class belong those texts which treat the subject rigorously and emphasize the real number system, limits, and continuity. Unfortunately, this is so time-consuming that applications are slighted. As a result, the student gains little ability to solve problems and frequently has no insight as to why the concepts he has learned are important.

The author of the present text has experimented, revised, experimented, and revised in the difficult task of merging the two approaches. I think the result is a very promising text that first emphasizes the uses of calculus, gradually exposes the analytical problems, and ultimately resolves them when the student has seen their importance.

I. M. Singer

Preface

Calculus has been the principal language of science ever since the seventeenth century, when it was invented, and it is likely to continue in this central role for some time to come. While Euclidean and analytic geometry give good descriptions of static figures in the plane and in space, calculus provides the means to study how things change; and it is the task of every scientific discipline to discover what laws govern the changes that take place around us. As calculus continues to contribute to the overwhelming development of physical science (and social science as well), it becomes more and more important to study and understand this subject, which is a language, a tool, and a logical discipline all in one.

The absolutely essential factor in understanding calculus (or anything else, for that matter) is thinking about it; the quality of books and teachers is secondary, except insofar as they can bring this about. I have tried to facilitate thinking by giving appropriate background and foreground material, by a scheme of organization that tries to discuss questions that are at least conceivably relevant to the student at the time he is reading about them, and by deliberately withholding answers to most of the problems. (Some problems, of course, give the answer away in their very statement.) I think that I understand the impulse that makes students prefer a problem whose answer is known, but this does not make me any more inclined to yield to the impulse. The classroom-and-textbook situation is artificial enough without divesting the problems of the essential feature of a real problem, namely that the solution is a priori unknown, and the solver must convince himself and his audience that he has really found it. A small number of problems worked in this spirit are worth ten times as many in which various computations are thrown into the gap between a given question and a given answer.

Aside from the problems which are offered as food for thought at the end of each section, there are simple exercises spread throughout the text to help the reader test his comprehension on the spot. If he finds an exercise that he cannot do, he should go back and reread. If he does the exercise, or if he is stuck even after a careful rereading of the material preceding it, then he should consult the solution which is given in the back of the book, either to check his own solution, or to see how he was supposed to do it.

Although the basic outline of the book is explained in §0.3, a few points should be made here.

I have tried to accommodate the rather wide range of preparation and ability that is found in most calculus classes, with the result that the beginning of the book will be too easy for most students, and the end will be too hard. The rigorous treatment of limits and integration is given in appendices at the end; whenever the reader finds himself dissatisfied with the mathematical discussions in the first half of the book, he should look to the corresponding discussion in the appendix. If the appendix makes sense to him at that point, fine; if not, he can carry on with the main text, and return to the appendix after he has been better prepared for it, particularly after Chapters VIII and IX.

Many texts achieve their aims by an ingeniously interwoven mosaic, presenting first a small part of each main topic, and then returning periodically to develop each one a little further. Even the rules for differentiation may be scattered, some coming quite early, and others (e.g. logarithms and inverse functions) rather late. This approach can be very effective when the instructor is thoroughly familiar with the text, but I feel that a great deal is lost in clarity of outline, and that there is bound to be some frustration in not pushing things through to their natural conclusion. I have found it more natural to develop each main topic systematically in a single chapter of its own (with the exception of differential equations, which arise at various points from Chapter V onward).

Any success in filling in this outline must be shared with my most patient and persistent critics, Hugo Rossi, I. M. Singer, and Nat Weintraub, with the instructors and students who struggled through the first version of this work, with all its inevitable (though largely unforeseen) difficulties, and with the publisher, who has been called upon for his share of patience and persistence. I hope that all these labors have not been in vain.

Robert T. Seeley

Contents

Introduction

The road to understanding in calculus is a long one. Before setting out, we would like to give a quick sketch of how calculus developed, and to introduce briefly some of the men who were responsible for this development (§0.1). Part of the development consisted of a subtle but important change in the foundations of calculus and its parent, analytic geometry; this is explained in §0.2. With this background, we outline the plan of the book in §0.3, and suggest several ways to use it.

0.1 A THUMBNAIL SKETCH OF THE HISTORY OF CALCULUS

Preliminary steps. Calculus rests on mathematical developments that go back as far as four thousand years to the Babylonians and the Egyptians, but we won't start there. The immediate contributions came at the end of the Renaissance.

First, there was the development of algebra. In the fifteen hundreds, the Italians achieved spectacular results in the solution of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, and in the process they advanced the use of negative and complex numbers. In 1585, Simon Stevin of Bruges published *La Disme*, the first proposal of a systematic use of decimal expansions. And there were improvements in notation by many, including René Descartes, abandoning clumsy verbal communication in favor of symbols very much like ours today. (This step is more important than it seems. A good notation not only saves space on the paper; it saves space in the memory and effort in the mind, both of which are at a premium.)

Second, there was the development of analytic geometry, due in large part to Descartes' *La Géometrie* (1637), which made it obvious that this new efficiency in algebra could be well employed in geometry. Pierre Fermat, a contemporary of Descartes, also made important contributions in this direction, finding equations of the straight lines and conic sections.

Third, there were various methods for determining tangents, areas, and volumes that led naturally to the ideas of differentiation and integration, the two main ideas of calculus. The best known of these methods are Fermat's way of drawing tangents (discovered about 1629) and B. Cavalieri's determination of areas and volumes, published in 1635. And Isaac Barrow showed an important connection between these two ideas, tangents and area, that was a direct forerunner of the fundamental theorem relating differentiation and integration.

Fourth, there were results on infinite series and products, primarily those of John Wallis, which appeared in 1655.

Finally, in physics there was a concerted study of motion, primarily the motion of falling bodies and of the planets. The most famous men in this work were Galileo, the astronomical observer Tycho Brahe, and Johann Kepler, who analyzed Brahe's observations and deduced three simple but profound empirical laws governing planetary motion.

The invention of calculus. Isaac Newton, a student of Barrow, gathered the mathematical developments together into one general theory, calculus, and applied it to solve the physical problems of the motion of falling bodies and of the planets. He calculated the orbit of a planet on the assumption that it was attracted by a force inversely proportional to the square of the distance from the sun (a rule that had been guessed at by others). The result agreed with Kepler's analysis!

Newton did not shout *Eureka!* and run into the streets to announce his discovery. Perhaps his caution was due to an error in the commonly accepted distance to the moon, which was not corrected until 1679: because of the error, the force of gravity at the surface of the earth (as found from falling bodies) did not agree well enough with the force that Newton derived to explain the motion of the moon. Whatever the cause of the delay, by the time his discoveries were finally made known, many of those having to do with calculus had already been found independently by his contemporary Gottfried Leibniz. Thus Newton and Leibniz are both considered the inventors of calculus.

Development and application. Neither Newton nor Leibniz succeeded in making the logic of their methods understood. Their reasoning was so mysterious that George Berkeley, an Irish bishop, published in 1734 the famous pamphlet *The Analyst* in which he defended the faith by pointing out that Newton and his followers treated objects no more substantial than "ghosts of departed quantities," and that the foundations of religion were every bit as secure as those of Newton's analysis.

In spite of the logical difficulties, both Newton and Leibniz had strong evidence that their methods contained some essential truth. Newton could explain the motion of the planets. And Leibniz had expressed his discoveries in a notation so apt that, although nobody understood exactly why, it led automatically to results that were seen to be correct.

From the end of the seventeenth century to the beginning of the nineteenth, calculus developed in the notation and outlook of Leibniz, but continued to find its inspiration and application in the project of explaining the physical world by mathematics, so successfully begun by Newton. The greatest mathematicians of this period were Leonhard Euler (1707–1783) and Joseph Louis Lagrange (1736–1813). Euler wrote the first widely read texts on calculus and others equally popular on algebra and trigonometry. He made advances in all fields of mathematics, and in dynamics, in the study of least action and energy, in the three-body problem of astronomy (the effect of mutual attractions between the earth, moon, and sun), in hydraulics, and in optics. Lagrange pursued these same questions, achieving greater unity and generality. His greatest work is the monumental *Mécanique analytique* (1788), which brought the science of mechanics close to its present form.

The great wealth of mathematical results, consistent with itself and with physical observations, proved beyond a doubt that calculus had abstracted certain essential features of the universe in which we live. But the logical foundations were still poorly understood, and even Euler was occasionally led by his formal manipulations to results that can hardly be considered correct. (One of these aberrations serves as a bad example in Chapter IX below. Unfortunately, most of Euler's work is beyond the scope of this book, and we are hardly able to balance the bad impression of this one example.)

The first great mathematician of the nineteenth century was Carl Friedrich Gauss (1777–1855), who made important contributions in the theory of the integers, use of infinite series, theory of surfaces, complex numbers, difficult numerical computations, astronomy, electricity and magnetism, surveying, and development of the telegraph.

Securing the foundations. A further contribution of Gauss was to the underlying logic of calculus. This development continued with Augustin Cauchy's book *Cours d'analyse* (1821) and culminated in the work of Karl Weierstrass (1815–1897) and Richard Dedekind (1831–1916). Dedekind's contribution was a penetrating analysis of the nature of the real numbers; Weierstrass pointed out subtle logical oversights in the work of his predecessors, and in his own work he adopted the standards of rigor and logic that still apply today.

The nature of the logical problems and the means of overcoming them are illustrated by an elementary geometric example in §0.2.

0.2 FROM ANALYTIC GEOMETRY TO ANALYTIC PROOFS

Analytic geometry employs algebraic methods to obtain geometric results. The link between algebra and geometry is a coordinate system, an idea that has been known for thousands of years. Two axes are drawn as in Fig. 1, and to each point P is assigned the ordered pair of numbers

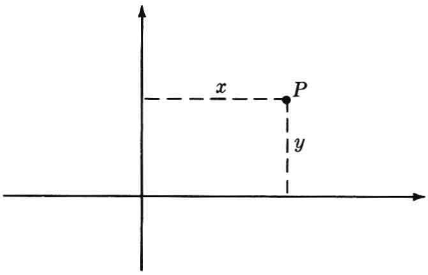

FIGURE 0.1

(x,y), x being the distance from P to one axis, and y the distance from P to the other axis. This process can be reversed: given x and y, you can measure off the corresponding distances from the axes and thus find P. Hence every statement about points can be translated into a corresponding statement about ordered pairs of numbers, and vice versa.

The correspondence is only valuable when algebra is well enough understood so that we can solve the algebraic translation of the geometric problem. By the time of Descartes, algebra was up to this challenge, and when he proposed as a general method the translation of problems from geometry to algebra, the idea was taken up so enthusiastically that there were actually complaints about the "clatter of the coordinate mill."

But for over a century there were no complaints about the underlying logic of analytic geometry. It was well based on the geometry of Euclid, and this was above reproach. However, in the course of soul-searching over the foundations of calculus, weaknesses were found even in Euclid.

One of the weak points shows up in Euclid's construction of an equilateral triangle on a given base. Take the given base AB as in Fig. 2.

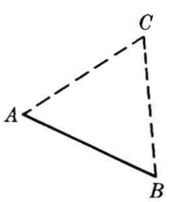

FIGURE 0.2

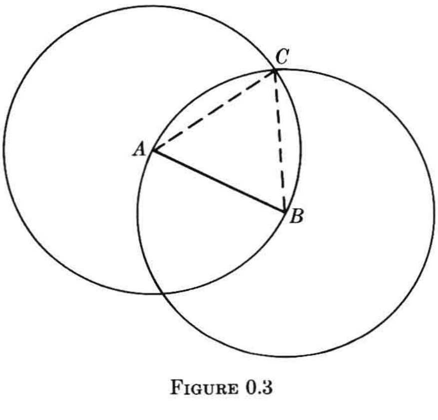

FIGURE 0.3

The problem is to show that there is a point C such that the lengths AC and BC equal the length of the given segment AB. The construction is done by drawing about each endpoint A and B a circle of radius $r = AB$, as in Fig. 3. Let C be a point of intersection of the two circles. Then BC and AC are radii of the two circles, so $BC = r = AB$ and $AC = r = AB$, and hence ABC is an equilateral triangle.

The picture is clear, but the proof is not complete. Euclid's axioms and postulates *do not guarantee* that the two circles will intersect, so this argument does not provide the desired point C.

But it is easy to prove by analytic geometry that there *is* a point C forming an equilateral triangle ABC. Let $A = (x_1,y_1)$ and $B = (x_2,y_2)$ be the given points, and let $C = (x,y)$ be any other point. Then, by the distance formula of analytic geometry, we have for the three distances AB, AC, BC,

$$\begin{aligned} AB &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \\ AC &= \sqrt{(x_1 - x)^2 + (y_1 - y)^2}, \\ BC &= \sqrt{(x_2 - x)^2 + (y_2 - y)^2}. \end{aligned} \tag{1}$$

If we set

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} + \sqrt{3}\,\frac{y_1 - y_2}{2}, \\ y &= \frac{y_1 + y_2}{2} + \sqrt{3}\,\frac{x_2 - x_1}{2}, \end{aligned} \tag{2}$$

it is a routine matter to check that the three distances in (1) are the same. Hence we have found the desired point C.

This simple example illustrates the most basic problem in calculus and geometry, the *existence* problem. Arguing geometrically, we inferred the existence of the point C by looking at the picture, which is not really legitimate. But the problem is easily solved in analytic geometry, for we deduce the existence of C from a certain property of numbers: for every positive number a, there exists a positive square root b such that $b^2 = a$. We use this property in writing the distance formulas (1) and in using $\sqrt{3}$ in the formulas (2) for x and y.

In calculus, besides this existence question, there is the problem of giving precise definitions of the subtle processes of differentiation and integration. These problems, too, can be solved by appeal to the properties of numbers. Thus we are led to the modern point of view, which is briefly this:

We start with a system of numbers, the real numbers (rationals and irrationals), which is developed without any appeal to geometry except, perhaps, in the role of interpreter.