T. SUNDARA ROW

GEOMETRIC
EXERCISES
IN PAPER
FOLDING

GEOMETRIC EXERCISES
IN PAPER FOLDING

by T. Sundara Row

Edited and Revised by
WOOSTER WOODRUFF BEMAN
and
DAVID EUGENE SMITH

EDITORS' PREFACE.

OUR attention was first attracted to Sundara Row's *Geometrical Exercises in Paper Folding* by a reference in Klein's *Vorlesungen über ausgewählte Fragen der Elementargeometrie*. An examination of the book, obtained after many vexatious delays, convinced us of its undoubted merits and of its probable value to American teachers and students of geometry. Accordingly we sought permission of the author to bring out an edition in this country, which permission was most generously granted.

The purpose of the book is so fully set forth in the author's introduction that we need only to say that it is sure to prove of interest to every wide-awake teacher of geometry from the graded school to the college. The methods are so novel and the results so easily reached that they cannot fail to awaken enthusiasm.

Our work as editors in this revision has been confined to some slight modifications of the proofs, some additions in the way of references, and the insertion of a considerable number of half-tone reproductions of actual photographs instead of the line-drawings of the original.

W. W. BEMAN.
D. E. SMITH.

INTRODUCTION.

THE *idea* of this book was suggested to me by Kindergarten Gift No. VIII.—Paper-folding. The gift consists of two hundred variously colored squares of paper, a folder, and diagrams and instructions for folding. The paper is colored and glazed on one side. The paper may, however, be of self-color, alike on both sides. In fact, any paper of moderate thickness will answer the purpose, but colored paper shows the creases better, and is more attractive. The kindergarten gift is sold by any dealers in school supplies ; but colored paper of both sorts can be had from stationery dealers. Any sheet of paper can be cut into a square as explained in the opening articles of this book, but it is neat and convenient to have the squares ready cut.
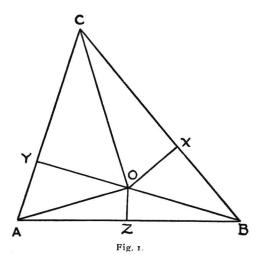
2. These exercises do not require mathematical instruments, the only things necessary being a penknife and scraps of paper, the latter being used for setting off equal lengths. The squares are themselves simple substitutes for a straight edge and a T square.

3. In paper-folding several important geometric processes can be effected much more easily than with

a pair of compasses and ruler, the only instruments
the use of which is sanctioned in Euclidean geom-
etry; for example, to divide straight lines and angles
into two or more equal parts, to draw perpendiculars
and parallels to straight lines. It is, however, not
possible in paper-folding to describe a circle, but a
number of points on a circle, as well as other curves,
may be obtained by other methods. These exercises
do not consist merely of drawing geometric figures
involving straight lines in the ordinary way, and fold-
ing upon them, but they require an intelligent appli-
cation of the simple processes peculiarly adapted to
paper-folding. This will be apparent at the very com-
mencement of this book.

4. The use of the kindergarten gifts not only affords
interesting occupations to boys and girls, but also
prepares their minds for the appreciation of science
and art. Conversely the teaching of science and art
later on can be made interesting and based upon
proper foundations by reference to kindergarten occu-
pations. This is particularly the case with geometry,
which forms the basis of every science and art. The
teaching of plane geometry in schools can be made
very interesting by the free use of the kindergarten
gifts. It would be perfectly legitimate to require pu-
pils to fold the diagrams with paper. This would
give them neat and accurate figures, and impress the
truth of the propositions forcibly on their minds. It
would not be necessary to take any statement on trust.

But what is now realised by the imagination and ideal-
isation of clumsy figures can be seen in the concrete.
A fallacy like the following would be impossible.

5. *To prove that every triangle is isosceles.* ' Let
ABC, Fig. 1, be any triangle. Bisect *AB* in *Z*, and
through *Z* draw *ZO* perpendicular to *AB*. Bisect the
angle *ACB* by *CO*.

Fig. 1.

(1) If *CO* and *ZO* do not meet, they are parallel.
Therefore *CO* is at right angles to *AB*. Therefore
AC = *BC*.

(2) If *CO* and *ZO* do meet, let them meet in *O*.
Draw *OX* perpendicular to *BC* and *OY* perpendicular
to *AC*. Join *OA*, *OB*. By Euclid I, 26 (B. and S.,
§ 88, cor. 7)* the triangles *YOC* and *XOC* are con-

*These references are to Beman and Smith's *New Plane and Solid Geom-
etry*, Boston, Ginn & Co., 1899.

gruent; also by Euclid I, 47 and I, 8 (B. and S., §
156 and § 79) the triangles AOY and BOX are con-
gruent.   Therefore

$$AY + YC = BX + XC,$$
$$\text{i. e., } AC = BC.$$

Fig. 2 shows by paper-folding that, whatever tri-
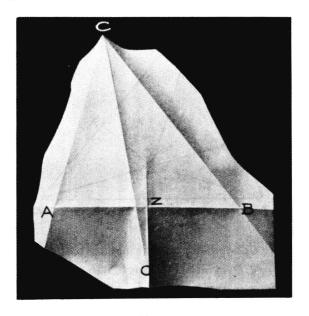angle be taken, CO and ZO cannot meet within the
triangle.

Fig. 2.

O is the mid-point of the arc AOB of the circle
which circumscribes the triangle ABC.

6. Paper-folding is not quite foreign to us.   Fold-
ing paper squares into natural objects—a boat, double

boat, ink bottle, cup-plate, etc., is well known, as also the cutting of paper in symmetric forms for purposes of decoration. In writing Sanskrit and Mahrati, the paper is folded vertically or horizontally to keep the lines and columns straight. In copying letters in public offices an even margin is secured by folding the paper vertically. Rectangular pieces of paper folded double have generally been used for writing, and before the introduction of machine-cut letter paper and envelopes of various sizes, sheets of convenient size were cut by folding and tearing larger sheets, and the second half of the paper was folded into an envelope inclosing the first half. This latter process saved paper and had the obvious advantage of securing the post marks on the paper written upon. Paper-folding has been resorted to in teaching the XIth Book of Euclid, which deals with figures of three dimensions.* But it has seldom been used in respect of plane figures.

7. I have attempted not to write a complete trea tise or text-book on geometry, but to show how regular polygons, circles and other curves can be folded or pricked on paper. I have taken the opportunity to introduce to the reader some well known problems of ancient and modern geometry, and to show how algebra and trigonometry may be advantageously applied to geometry, so as to elucidate each of the subjects which are usually kept in separate pigeon-holes.

* See especially Beman and Smith's *New Plane and Solid Geometry*, p. 287.

8. The first nine chapters deal with the folding of the regular polygons treated in the first four books of Euclid, and of the nonagon. The paper square of the kindergarten has been taken as the foundation, and the other regular polygons have been worked out thereon. Chapter I shows how the fundamental square is to be cut and how it can be folded into equal right-angled isosceles triangles and squares. Chapter II deals with the equilateral triangle described on one of the sides of the square. Chapter III is devoted to the Pythagorean theorem (B. and S., § 156) and the propositions of the second book of Euclid and certain puzzles connected therewith. It is also shown how a right-angled triangle with a given altitude can be described on a given base. This is tantamount to finding points on a circle with a given diameter.

9. Chapter X deals with the arithmetic, geometric, and harmonic progressions and the summation of certain arithmetic series. In treating of the progressions, lines whose lengths form a progressive series are obtained. A rectangular piece of paper chequered into squares exemplifies an arithmetic series. For the geometric the properties of the right-angled triangle, that the altitude from the right angle is a mean proportional between the segments of the hypotenuse (B. and S., § 270), and that either side is a mean proportional between its projection on the hypotenuse and the hypotenuse, are made use of. In this connexion the Delian problem of duplicating a cube has been

explained.* In treating of harmonic progression, the fact that the bisectors of an interior and corresponding exterior angle of a triangle divide the opposite side in the ratio of the other sides of the triangle (B. and S., § 249) has been used. This affords an interesting method of graphically explaining systems in involution. The sums of the natural numbers and of their cubes have been obtained graphically, and the sums of certain other series have been deduced therefrom.

10. Chapter XI deals with the general theory of regular polygons, and the calculation of the numerical value of π. The propositions in this chapter are very interesting.

11. Chapter XII explains certain general principles, which have been made use of in the preceding chapters,—congruence, symmetry, and similarity of figures, concurrence of straight lines, and collinearity of points are touched upon.

12. Chapters XIII and XIV deal with the conic sections and other interesting curves. As regards the circle, its harmonic properties among others are treated. The theories of inversion and co-axial circles are also explained. As regards other curves it is shown how they can be marked on paper by paperfolding. The history of some of the curves is given, and it is shown how they were utilised in the solution

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* See Beman and Smith's translation of Klein's *Famous Problems of Elementary Geometry*, Boston, 1897; also their translation of Fink's *History of Mathematics*, Chicago, The Open Court Pub. Co., 1900.

of the classical problems, to find two geometric means between two given lines, and to trisect a given rectilineal angle. Although the investigation of the properties of the curves involves a knowledge of advanced mathematics, their genesis is easily understood and is interesting.

13. I have sought not only to aid the teaching of geometry in schools and colleges, but also to afford mathematical recreation to young and old, in an attractive and cheap form. "Old boys" like myself may find the book useful to revive their old lessons, and to have a peep into modern developments which, although very interesting and instructive, have been ignored by university teachers.

T. SUNDARA ROW.

MADRAS, INDIA, 1893.

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