P ADAMS
K SMITH
R VÝBORNÝ

INTRODUCTION TO MATHEMATICS WITH MAPLE

P ADAMS
K SMITH
R VÝBORNÝ
The University of Queensland, Australia

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INTRODUCTION TO MATHEMATICS WITH MAPLE

To
Narelle, Helen and Aťa
for their understanding and support
while we were writing this book.

Preface

> I attempted mathematics, and even went during the summer of 1828 with a private tutor to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense.
>
> Charles Darwin, *Autobiography* (1876)

Charles Darwin wasn't the last biologist to regret not knowing more about mathematics. Perhaps if he had started his university education at the beginning of the 21^{st} century, instead of in 1828, and had been able to profit from using a computer algebra package, he would have found the material less forbidding.

This book is about pure mathematics. Our aim is to equip the readers with understanding and sufficiently deep knowledge to enable them to use it in solving problems. We also hope that this book will help readers develop an appreciation of the intrinsic beauty of the subject!

We have said that this book is about mathematics. However, we make extensive use of the computer algebra package *Maple* in our discussion. Many books teach pure mathematics without any reference to computers, whereas other books concentrate too heavily on computing, without explaining substantial mathematical theory. We aim for a better balance: we present material which requires deep thinking and understanding, but we also fully encourage our readers to use *Maple* to remove some of the laborious computations, and to experiment. To this end we include a large number of *Maple* examples.

Most of the mathematical material in this book is explained in a fairly traditional manner (of course, apart from the use of *Maple*!). However, we depart from the traditional presentation of integral by presenting the Kurzweil—Henstock theory in Chapter 15.

Outline of the book

There are fifteen chapters. Each starts with a short abstract describing the content and aim of the chapter.

In Chapter 1, "Introduction", we explain the scope and guiding philosophy of the present book, and we make clear its logical structure and the role which *Maple* plays in the book. It is our aim to equip the readers with sufficiently deep knowledge of the material presented so they can use it in solving problems, and appreciate its inner beauty.

In Chapter 2, "Sets", we review set theoretic terminology and notation and provide the essential parts of set theory needed for use elsewhere in the book. The development here is not strictly axiomatic—that would require, by itself, a book nearly as large as this one—but gives only the most important parts of the theory. Later we discuss mathematical reasoning, and the importance of rigorous proofs in mathematics.

In Chapter 3, "Functions", we introduce relations, functions and various notations connected with functions, and study some basic concepts intimately related to functions.

In Chapter 4, "Real Numbers", we introduce real numbers on an axiomatic basis, solve inequalities, introduce the absolute value and discuss the least upper bound axiom. In the concluding section we outline an alternative development of the real number system, starting from Peano's axioms for natural numbers.

In Chapter 5, "Mathematical Induction", we study proof by induction and prove some important inequalities, particularly the arithmetic-geometric mean inequality. In order to employ induction for defining new objects we prove the so-called recursion theorems. Basic properties of powers with rational exponents are also established in this chapter.

In Chapter 6, "Polynomials", we introduce polynomials. Polynomial functions have always been important, if for nothing else than because, in the past, they were the only functions which could be readily evaluated. In this chapter we define polynomials as algebraic entities rather than functions and establish the long division algorithm in an abstract setting. We also look briefly at zeros of polynomials and prove the Taylor Theorem for polynomials in a generality which cannot be obtained by using methods of calculus.

In Chapter 7, "Complex Numbers", we introduce complex numbers, that is, numbers of the form $a + b\imath$ where the number $\imath$ satisfies $\imath^2 = -1$. Mathematicians were led to complex numbers in their efforts of solving algebraic equations, that is, of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, with the a_k real numbers and n a positive integer (this problem is perhaps more widely known as finding the zeros of a polynomial). Our introduction follows the same idea although in a modern mathematical setting. Complex numbers now play important roles in physics, hydrodynamics, electromagnetic theory and electrical engineering, as well as pure mathematics.

In Chapter 8, "Solving Equations", we discuss the existence and uniqueness of solutions to various equations and show how to use *Maple* to find solutions. We deal mainly with polynomial equations in one unknown, but include some basic facts about systems of linear equations.

In Chapter 9, "Sets Revisited", we introduce the concept of equivalence for sets and study countable sets. We also briefly discuss the axiom of choice.

In Chapter 10, "Limits of Sequences", we introduce the idea of the limit of a sequence and prove basic theorems on limits. The concept of a limit is central to subsequent chapters of this book. The later sections are devoted to the general principle of convergence and more advanced concepts of limits superior and limits inferior of a sequence.

In Chapter 11, "Series", we introduce infinite series and prove some basic convergence theorems. We also introduce power series—a very powerful tool in analysis.

In Chapter 12, "Limits and Continuity of Functions", we define limits of functions in terms of limits of sequences. With a function f continuous on an interval we associate the intuitive idea of the graph f being drawn without lifting the pencil from the drawing paper. The mathematical treatment of continuity starts with the definition of a function continuous at a point; this definition is given here in terms of a limit of a function at a point. We develop the theory of limits of functions, study continuous functions, and particularly functions continuous on closed bounded intervals. At the end of the chapter we touch upon the concept of limit superior and inferior of a function.

In Chapter 13, "Derivatives", we start with the informal description of a derivative as a rate of change. This concept is extremely important in science and applications. In this chapter we introduce derivatives as limits, establish their properties and use them in studying deeper properties of functions and their graphs. We also extend the Taylor Theorem from polynomials to power series and explore it for applications.

In Chapter 14, "Elementary Functions", we lay the proper foundations for the exponential and logarithmic functions, and for trigonometric functions and their inverses. We calculate derivatives of these functions and use these for establishing important properties of these functions.

In Chapter 15, "Integrals", we present the theory of integration introduced by the contemporary Czech mathematician J. Kurzweil. Sometimes it is referred to as Kurzweil—Henstock theory. Our presentation generally follows Lee and Výborný (2000, Chapter 2).

The Appendix contains some examples of *Maple* programs. Finally, the book concludes with a list of References, an Index of *Maple* commands used in the book, and a general Index.

Notes on notation

Throughout the book there are a number of ways in which the reader's attention is drawn to particular points. Theorems, lemmas[1] and corollaries are placed inside rectangular boxes with double lines, as in

> **Theorem 0.1 (For illustrative purposes only!)** *This theorem is referred to only in the* Preface, *and can safely be ignored when reading the rest of the book.*

Note that these are set in slanting font, instead of the upright font used in the bulk of the book.

Definitions are set in the normal font, and are placed within rectangular boxes, outlined by a single line and with rounded corners, as in

> **Definition 0.1 (What is mathematics?)** There are almost as many definitions of what mathematics is as there are professional mathematicians living at the time.

[1]A lemma is sometimes known as an auxiliary theorem. It does not have the same level of significance as a theorem, and is usually proved separately to simplify the proof of the related theorem(s).

The number before the decimal point in all of the above is the number of the chapter: numbers following the decimal point label the different theorems, definitions, examples, etc., and are numbered consecutively (and separately) within each chapter. Corollaries are labeled by the number of the chapter, followed by the number of the theorem to which the corollary belongs then followed by the number of the individual corollary for that theorem, as in

> **Corollary 0.1.1 (Also for illustrative purposes only!)** *Since* Theorem 0.1 *is referred to only in the* Preface, *any of its corollaries can also be ignored when reading the rest of the book.*

> **Corollary 0.1.2 (Second corollary for Theorem 0.1)** *This is just as helpful as the first corollary for the theorem!*

Corollaries are set in the same font as theorems and lemmas.

Most of the theorems, lemmas and corollaries in this book are provided with proofs. All proofs commence with the word "***Proof.***" flush with the left margin. Since the words of a proof are set in the same font as the rest of the book, a special symbol is used to mark the end of a proof and the resumption of the main text. Instead of saying that a proof is complete, or words to that effect, we shall place the symbol □ at the end of the proof, and flush with the right margin, as follows:

Proof. This is not really a proof. Its main purpose is to illustrate the occurrence of a small hollow square, flush with the right margin, to indicate the end of a proof. □

Up to about the middle of the 20^{th} century it was customary to use the letters 'q.e.d.' instead of □, q.e.d. being an abbreviation for *quod erat demonstrandum*, which, translated from Latin, means *which was to be proved.*

To assist the reader, there are a number of Remarks scattered throughout the book. These relate to the immediately preceding text. There are also Examples of various kinds, used to illustrate a concept by providing a (usually simple) case which can show the main distinguishing points of a concept. Remarks and Examples are set in sans serif font, like this, to help distinguish them.

Remark 0.1 The first book published on calculus was Sir Isaac Newton's *Philosophiae Naturalis Principia Mathematica* (Latin for *The Mathematical Principles of Natural Philosophy*), commonly referred to simply as *Principia*. As might be expected from the title, this was in Latin. We shall avoid the use of languages other than English in this book.

Since they use a different font, and have additional spacing above and below, it is obvious where the end of a Remark or an Example occurs, and no special symbol is needed to mark the return to the main text.

Scissors in the margin

Obviously, we must build on some previous knowledge of our readers. Chapter 1 and Chapter 2 summarise such prerequisites. Chapter 1 contains also a brief introduction to *Maple*. Starting with Chapter 3 we have tried to make sure that all proofs are in a strict logical order. On a few occasions we relax the logical requirements in order to illustrate some point or to help the reader place the material in a wider context. All such instances are clearly marked in the margin (see the outer margin of this page), the beginning by scissors pointing into the book, the end by scissors opening outwards. The idea here is to indicate readers can skip over these sections if they desire strict logical purity. For instance, we might use trigonometric functions before they are properly introduced,[2] but then this example will be scissored.

Exercises

There are exercises to help readers to master the material presented. We hope readers will attempt as many as possible. Mathematics is learned by doing, rather than just reading. Some of the exercises are challenging, and these are marked in the margin by the symbol (!). We do not expect that readers will make an effort to solve all these challenging problems, but should attempt at least some. Exercises containing fairly important additional information, not included in the main body of text, are marked by (i). We recommend that these should be read even if no attempt is made to solve them.

[2] Rather late in Chapter 14

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Our greatest debt is to our wives. As a small token of our love and appreciation we dedicate this book to them.

Peter Adams
Ken Smith
Rudolf Výborný

The University of Queensland, February 2004.

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