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Algebraic Topology

Ched E. Stedman

Editor

ALGEBRA AND ALGEBRAIC TOPOLOGY

CHED E. STEDMAN
EDITOR

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ALGEBRA AND ALGEBRAIC TOPOLOGY

PREFACE

This new volume gathers results in pure and applied algebra including algebraic topology from researchers around the globe. The selection of these papers was carried out under the auspices of a special editorial board.

In Chapter 1 the authors compute the Chen-Ruan cohomology rings of the weighted projective spaces, a class of important spaces in algebraic geometry and physics. The classical tools (Chen-Ruan cohomology, toric varieties, the localization technique) which have been proved to be successful are used to study the orbifold cohomology of weighted projective spaces. Given a weighted projective space $P^n_{q_0,\ldots,q_n}$, the authors determine all of its twisted sectors and the corresponding degree shifting numbers, and they calculate the orbifold cohomology group of $P^n_{q_0,\ldots,q_n}$. For a general reduced weighted projective space, they determine the obstruction bundle over any 3-multisector and give a formula to compute the 3-point function which is the key in the definition of Chen-Ruan cohomology ring. Finally they concretely calculate the Chen-Ruan cohomology ring of weighted projective space $P^5_{1,2,2,3,3,3}$

James showed that a space X is an *H*-space if and only if there is a retraction r : $\Omega\Sigma X \to X$. Then Stasheff showed that there is an H-map retraction of X if and only if the multiplication of X is homotopy associative. Hemmi generalized this result to a theorem for A_n-spaces. In Chapter 2 the authors first study the relations between the structure of the multiplication of an *H*-space and the one of retractions of the *H*-space in more detail. Then we extend the results in [2] to a theorem for maps $r_i : \Omega P_iX \to X$ $(1 \leq i \leq n-1)$ for an A_n-space X, where P_iX is the projective i-space of X with $P_1X = \Sigma X$.

The best least squares fit Λ_A to a matrix A in a space Λ can be useful to improve the rate of convergence of the conjugate gradient method in solving systems $Ax = b$ as well as to define low complexity quasi-Newton algorithms in unconstrained minimization. This is shown in Chapter 3 with new important applications and ideas. Moreover, some theoretical results on the representation and on the computation of Λ_A are investigated.

A pair of sign pattern row vectors (respectively, column vectors) allows orthogonality if the two vectors are the sign patterns of two real orthogonal row vectors (respectively, column vectors). A square sign pattern matrix that does not have a zero row or zero column is sign potentially orthogonal (SPO) if every pair of rows and every pair of columns allows orthogonality. In Chapter 4, the authors prove that when n is even, there is a k-regular SPO sign pattern of order n if and only if $1 \leq k \leq n$; when n 6= 5 is odd, there is a k-regular SPO sign pattern of order n if and only if $1 \leq k \leq n$ and k 6= 2; when n = 5, there is a k-regular SPO sign pattern of order n if and only if k 6= 2 and k 6= 3.

Partition problems are classical problems of the combinatorial geometry whose solutions often rely on the methods of the equivariant topology. The *k*-fan partition problems introduced in [11] and first discussed by equivariant methods in [2], [3] have forced some hard concrete combinatorial calculations in equivariant cohomology [5], [4]. These problems can be reduced, by the beautiful scheme of Bárány andMatoušek, [2], to topological problems of the existence of Δ_{2n} equivariant maps $V_2(\mathbb{R}^3) \to W_n - \cup A(\alpha)$ from a Stiefel manifold of all orthonormal 2-frames in R_3 to complements of appropriate arrangements.

In Chapter 5 the authors e present a set of techniques, based on the equivariant obstruction theory, which can help in answering the question of the existence of a equivariant map to a complement of an arrangement. With the help of the target extension scheme, introduced in [5], they are able to deal with problems where the existence of the map depends on more then one obstruction. The introduced techniques, with an emphasis on computation, are applied on the known results of the fan partition problems.

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Chapter 1

THE CHEN-RUAN COHOMOLOGY OF WEIGHTED PROJECTIVE SPACES

Yunfeng Jiang*
Department of Mathematics, the University of British Columbia,
1984 Mathematics Rd, Vancouver, BC, V6T 1Z2, Canada

Abstract

We compute the Chen-Ruan cohomology rings of the weighted projective spaces, a class of important spaces in algebraic geometry and physics. The classical tools (Chen-Ruan cohomology, toric varieties, the localization technique) which have been proved to be successful are used to study the orbifold cohomology of weighted projective spaces. Given a weighted projective space $\mathbf{P}^n_{q_0,\cdots,q_n}$, we determine all of its twisted sectors and the corresponding degree shifting numbers, and we calculate the orbifold cohomology group of $\mathbf{P}^n_{q_0,\cdots,q_n}$. For a general reduced weighted projective space, we determine the obstruction bundle over any 3-multisector and give a formula to compute the 3-point function which is the key in the definition of Chen-Ruan cohomology ring. Finally we concretely calculate the Chen-Ruan cohomology ring of weighted projective space $\mathbf{P}^5_{1,2,2,3,3,3}$.

Key Words: Chen-Ruan cohomology, twisted sectors, toric varieties, weighted projective space, localization

1 Introduction

The notion of Chen-Ruan orbifold cohomology has appeared in physics as a result of studying the string theory on global quotient orbifold, (see [10] and [11]). In addition to the usual cohomology of the global quotient, this space included the cohomology of so-called twisted sectors. Zaslow [25] gave a lot of examples of global quotients and computed their orbifold cohomology spaces. But the real mathematical definition of orbifold cohomology was given by Chen and Ruan [6] for arbitrary orbifolds. The most interesting feature of this new cohomology theory, besides the generalization of non global quotients, is the existence of a ring

*E-mail address: jiangyf@math.ubc.ca

structure which was previously missing. This ring structure is obtained from Chen-Ruan's orbifold quantum cohomology construction (see [7]) by restricting to the class called ghost maps, the same as the ordinary cup product may be obtained by quantum cup product. Since the Chen-Ruan cohomology appeared, the problem of how to calculate the orbifold cohomology has been considered by several authors. Chen and Ruan [6] gave several simple examples. Chen Hao [4] computed the orbifold cohomology group of moduli space $\mathcal{M}_{0,n}/S_n$. B.Doug Park and Mainak Poddar [22] considered the Chen-Ruan cohomology ring of the mirror quintic. All the above examples are orbifold global quotients. In this paper we calculate the Chen-Ruan cohomology rings of weighted projective spaces-a large class of non-global quotient orbifolds.

A very power tool to compute the Chen-Ruan cohomology of weighted projective spaces is the method of toric varieties. The theory of toric varieties establishes a classical connection between algebraic geometry and the theory of convex polytopes. From the fan of a toric variety, we can obtain a lot of information about the toric variety. In particular, when the fan Σ of a toric variety X is simplicial, the toric variety X is an orbifold with finite abelian groups as local groups. In this paper, we take the weighted projective spaces as simplicial toric varieties with local isotropy groups the finite cyclic groups. And then using the properties of toric varieties induced from the fans, we calculate the Chen-Ruan cohomology group of any weighted projective space.

To calculate the Chen-Ruan cohomology ring of the weighted projective space, we use the Riemann bilinear relations for periods [15] to identify the obstruction bundle. Up to now, except that the obstruction bundle for the mirror quintic example calculated by Park and Poddar [22] is a nontrivial line bundle, all the other calculated obstruction bundles of the examples are trivial. In this paper, the obstruction bundle we consider is the Whitney sum of some line bundles, generalizing the case of mirror quintic example. And we also introduce the localization techniques [3] which should work for toric varieties to compute the 3-point function which is the key in the orbifold cup product [6]. In particular, we give a concrete example.

On the other hand, a very interesting aspect of calculating the Chen-Ruan cohomology rings of weighted projective spaces lies in a conjecture of Ruan. In string theory, physicists suggest that the orbifold string theory of an orbifold should be equivalent to the ordinary string theory of its crepant resolution. For the orbifold cohomology, the Cohomology Hyperkahler Resolution Conjecture of Ruan (see [23]) states that the Chen-Ruan cohomology ring of an orbifold should be isomorphic to the ordinary cohomology ring of its hyperkahler resolution. I hope that my calculation of the Chen-Ruan cohomology ring of the weighted projective space may contribute to this interesting problem.

The thesis is outlined as follows. Section 2 is a review of some basic facts concerning orbifold, Chen-Ruan cohomology and simplicial toric varieties. In section 3 we introduced the basic concept of the weighted projective space. In section 4 we discuss the Chen-Ruan cohomology group of any weighted projective space. And in the section 5 we compute the ring structure of the Chen-Ruan cohomology of the weighted projective space.

2 Preliminaries

2.1 Orbifold and Orbifold Vector Bundle

Definition 2.1.1. An orbifold structure on a Hausdorff, separate topological space X is given by an open cover $\mathcal{U}$ of X satisfying the following conditions.

(1) Each element U in $\mathcal{U}$ is uniformized, say by (V, G, π). Namely, V is a smooth manifold and G is a finite group acting smoothly on V such that $U = V/G$ with π as the quotient map. Let $Ker(G)$ be the subgroup of G acting trivially on V.

(2) For $U^{'} \subset U$, there is a collection of injections $(V^{'}, G^{'}, \pi^{'}) \longrightarrow (V, G, \pi)$. Namely, the inclusion $i : U^{'} \subset U$ can be lifted to maps $\widetilde{i} : V^{'} \longrightarrow V$ and an injective homomorphism $i_* : G^{'} \longrightarrow G$ such that i_* is an isomorphism from $Ker(G^{'})$ to $Ker(G)$ and $\widetilde{i}$ is i_*-equivariant.

(3) For any point $x \in U_1 \cap U_2$, $U_1, U_2 \in \mathcal{U}$, there is a $U_3 \in \mathcal{U}$ such that $x \in U_3 \subset U_1 \cap U_2$.

For any point $x \in X$, suppose that (V, G, π) is a uniformizing neighborhood and $\overline{x} \in \pi^{-1}(x)$. Let G_x be the stabilizer of G at $\overline{x}$. Up to conjugation, it is independent of the choice of $\overline{x}$ and is called the *local group* of x. Then there exists a sufficiently small neighborhood V_x of $\overline{x}$ such that (V_x, G_x, π_x) uniformizes a small neighborhood of x, where π_x is the restriction $\pi \mid V_x$. (V_x, G_x, π_x) is called *a local chart* at x. The orbifold structure is called *reduced* if the action of G_x is effective for every x.

Let $pr : E \longrightarrow X$ be a rank k complex *orbifold bundle* over an orbifold X([6]). Then a uniformizing system for $E \mid U = pr^{-1}(U)$ over a uniformized subset U of X consists of the following data:

(1) A uniformizing system (V, G, π) of U.

(2) A uniformizing system $(V \times \mathbf{C^k}, G, \widetilde{\pi})$ for $E \mid U$. The action of G on $V \times \mathbf{C}^k$ is an extension of the action of G on V given by $g \cdot (x, v) = (g \cdot x, \rho(x, g)v)$ where $\rho : V \times G \longrightarrow Aut(\mathbf{C}^k)$ is a smooth map satisfying:

$$\rho(g \cdot x, h) \circ \rho(x, g) = \rho(x, hg), g, h \in G, x \in V.$$

(3)The natural projection map $\widetilde{pr} : V \times \mathbf{C^k} \longrightarrow \mathbf{V}$ satisfies $\pi \circ \widetilde{pr} = pr \circ \widetilde{\pi}$.

By an orbifold connection $\triangle$ on E we mean an equivariant connection that satisfies $\triangle = g^{-1} \triangle g$ for every uniformizing system of E. Such a connection can be always obtained by averaging an equivariant partition of unity.

2.2 Twisted Sectors and Chen-Ruan Cohomology

The most physical idea is twisted sectors. Let X be an orbifold. Consider the set of pairs:

$$\widetilde{X}_k = \{(p, (\mathbf{g})_{G_p}) | p \in X, \mathbf{g} = (g_1, \cdots, g_k), g_i \in G_p\}$$

where $(\mathbf{g})_{G_p}$ is the conjugacy class of k-tuple $\mathbf{g} = (g_1, \cdots, g_k)$ in G_p. We use G^k to denote the set of k-tuples. If there is no confusion, we will omit the subscript G_p to simplify the notation. Suppose that X has an orbifold structure $\mathcal{U}$ with uniformizing systems $(\widetilde{U}, G_U, \pi_U)$. From Chen and Ruan [6], also see [18], we have: $\widetilde{X}_k$ is naturally an orbifold, with the generalized orbifold structure at $(p, (\mathbf{g})_{G_p})$ given by $(V_p^{\mathbf{g}}, C(\mathbf{g}), \pi : V_p^{\mathbf{g}} \longrightarrow V_p^{\mathbf{g}}/C(\mathbf{g}))$, where

$V_p^{\mathbf{g}} = V_p^{g_1} \cap \cdots V_p^{g_k}$, $C(\mathbf{g}) = C(g_1) \cap \cdots C(g_k)$. Here $\mathbf{g} = (g_1, \cdots, g_k)$. V_p^g stands for the fixed point set of g in V_p. When X is almost complex, $\widetilde{X_k}$ inherits an almost complex structure from X, and when X is closed, $\widetilde{X_k}$ is finite disjoint union of closed orbifolds.

Now we describe the the connected components of $\widetilde{X_k}$. Recall that every point p has a local chart (V_p, G_p, π_p) which gives a local uniformized neighborhood $U_p = \pi_p(V_p)$. If $q \in U_p$, up to conjugation there is a unique injective homomorphism $i_* : G_q \longrightarrow G_p$. For $\mathbf{g} \in (G_q)^k$, the conjugation class $i_*(\mathbf{g})_q$ is well defined. We define an equivalence relation $i_*(\mathbf{g})_q \cong (\mathbf{g})_q$. Let T_k denote the set of equivalence classes.To abuse the notation, we use $(\mathbf{g})$ to denote the equivalence class which $(\mathbf{g})_q$ belongs to. We will usually denote an element of T_1 by (g). It is clear that $\widetilde{X_k}$ can be decomposed as a disjoint union of connected components:

$$\widetilde{X_k} = \bigsqcup_{(\mathbf{g}) \in T_k} X_{(\mathbf{g})}$$

Where $X_{(\mathbf{g})} = \{(p, (\mathbf{g}')_p) | \mathbf{g}' \in (G_p)^k, (\mathbf{g}')_p \in (\mathbf{g})\}$. Note that for $\mathbf{g} = (1, \cdots, 1)$, we have $X_{(\mathbf{g})} = X$. A component $X_{(\mathbf{g})}$ is called a $k-multisector$, if $\mathbf{g}$ is not the identity. A component of $X_{(g)}$ is simply called a *twisted sector*. If X has an almost complex, complex or kahler structure, then $X_{(\mathbf{g})}$ has the analogous structure induced from X. We define

$$T_3^0 = \{(\mathbf{g}) = (g_1, g_2, g_3) \in T_3 | g_1 g_2 g_3 = 1\}.$$

Note that there is an one to one correspondence between T_2 and T_3^0 given by $(g_1, g_2) \longmapsto (g_1, g_2, (g_1 g_2)^{-1})$.

Now we define the Chen-Ruan cohomology. Assume that X is a n-dimensional compact almost complex orbifold with almost structure J. Then for a point p with nontrivial group G_p, J gives rise to an effective representation $\rho_p : G_p \longrightarrow GL(n, \mathbf{C})$. For any $g \in G_p$, we write $\rho_p(g)$, up to conjugation, as a diagonal matrix

$$diag\left(e^{2\pi i \frac{m_{1,g}}{m_g}}, \ldots, e^{2\pi i \frac{m_{n,g}}{m_g}}\right).$$

where m_g is the order of g in G_p, and $0 \leq m_{i,g} < m_g$. Define a function $\iota : \widetilde{X_1} \longrightarrow \mathbf{Q}$ by

$$\iota(p, (g)_p) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}.$$

We can see that the function $\iota : \widetilde{X_1} \longrightarrow \mathbf{Q}$ is locally constant and $\iota = 0$ if $g = 1$. Denote its value on $X_{(g)}$ by ι_g. We call ι_g the degree shifting number of $X_{(g)}$. It has the following properties:

(1) $\iota_{(g)}$ is an integer iff $\rho_p(g) \in SL(n, \mathbf{C})$;

(2) $\iota_{(g)} + \iota_{(g^{-1})} = rank(\rho_p(g) - Id) = n - dim_{\mathbf{C}} X_{(g)}$.

A C^∞ differential form on X is a G-invariant differential form on V for each uniformizing system (V, G, π). Then orbifold integration is defined as follows. Suppose $U = V/G$ is connected, for any compactly supported differential n-form ω on U, which is, by definition, a G-invariant n-form $\widetilde{\omega}$ on V,

$$\int_U^{orb} \omega := \frac{1}{|G|}\int_V \widetilde{\omega} \tag{2.1}$$

Where $|G|$ is the order of G. The orbifold integration over X is defined by using a C^∞ partition of unity. The orbifold integration coincides with the usual measure theoretic integration iff the orbifold structure is reduced.

Holomorphic forms for a complex orbifold X are again obtained by patching G-invariant holomorphic forms on the uniformizing system (V, G, π). We consider the Cech cohomology groups of X and $X_{(\mathbf{g})}$ with coefficients in the sheaves of holomorphic forms. The Cech cohomology groups can be identified with the Dolbeault cohomology groups of (p, q)-forms [2].

Definition 2.2.1. ([6]) Let X be a closed complex orbifold, we define the orbifold cohomology group of X by

$$H^d_{orb} := \bigoplus_{(g)\in T_1} H^{d-2\iota(g)}(X_{(g)}, \mathbf{Q})$$

For $0 \le p, q \le dim_{\mathbf{C}}X$, we define the orbifold Dolbeault cohomology group of X by

$$H^{p,q}_{orb}(X) := \bigoplus_{(g)\in T_1} H^{p-\iota(g),q-\iota(g)}(X_{(g)}, \mathbf{C})$$

2.3 The Obstruction Bundle

Choose $(\mathbf{g})= (g_1, g_2, g_3) \in T_3^0$. Let $(p, (\mathbf{g})_p)$ be a generic point in $X_{(\mathbf{g})}$. Let $K(\mathbf{g})$ be the subgroup of G_p generated by g_1 and g_2. Consider an orbifold Riemann sphere with three orbifold points $(S^2, (p_1, p_2, p_3), (k_1, k_2, k_3))$. When there is no confusion, we will simply denote it by S^2. The orbifold fundamental group is:

$$\pi_1^{orb}(S^2) = \{\lambda_1, \lambda_2, \lambda_3 | \lambda_i^{k_i} = 1, \lambda_1\lambda_2\lambda_3 = 1\}$$

Where λ_i is represented by a loop around the marked p_i. There is a surjective homomorphism

$$\rho : \pi_1^{orb}(S^2) \longrightarrow K(\mathbf{g})$$

specified by mapping $\lambda_i \longmapsto g_i$. $Ker(\rho)$ is a finite-index subgroup of $\pi_1^{orb}(S^2)$. Let $\widetilde{\Sigma}$ be the orbifold universal cover of S^2. Let $\Sigma = \widetilde{\Sigma}/Ker(\rho)$. Then Σ is smooth, compact and $\Sigma/K(\mathbf{g}) = S^2$. The genus of Σ can be computed using Riemann Hurwitz formula for Euler characteristics of a branched covering, and turns out to be

$$g(\Sigma) = \frac{1}{2}(2 + |K(\mathbf{g})| - \Sigma_{i=1}^3 \frac{|K(\mathbf{g})|}{k_i}) \tag{2.2}$$

$K(\mathbf{g})$ acts holomorphically on Σ and hence $K(\mathbf{g})$ acts on $H^{0,1}(\Sigma)$. The "obstruction bundle" $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$ is constructed as follows. On the local chart $(V_p^{\mathbf{g}}, C(\mathbf{g}), \pi)$ of $X_{(\mathbf{g})}$, $E_{(\mathbf{g})}$ is given by $(TV_p \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})} \times V_p^{\mathbf{g}} \longrightarrow V_p^{\mathbf{g}}$, where $(TV_p \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}$ is the $K(\mathbf{g})$-invariant subspace. We define an action of $C(\mathbf{g})$ on $TV_p \otimes H^{0,1}(\Sigma)$, which is the usual one on TV_p and trivial on $H^{0,1}(\Sigma)$. The the action of $C(\mathbf{g})$ and $K(\mathbf{g})$ commute and $(TV_p \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}$ is invariant under $C(\mathbf{g})$. Thus we have obtained an action of $C(\mathbf{g})$ on $(TV_p \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})} \times V_p^{\mathbf{g}} \longrightarrow V_p^{\mathbf{g}}$, extending the usual one on $V_p^{\mathbf{g}}$. These trivializations fit together to define the bundle $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$. If we set $e : X_{(\mathbf{g})} \longrightarrow X$ to be the map given by $(p, (\mathbf{g})_p) \longmapsto p$, one may think of $E_{(\mathbf{g})}$ as $(e^*TX \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}$. The rank of $E_{(\mathbf{g})}$ is given by the formula [6]:

$$rank_{\mathbf{C}}(E_{(\mathbf{g})}) = dim_{\mathbf{C}}(X_{(\mathbf{g})}) - dim_{\mathbf{C}}(X) + \Sigma_{j=1}^{3}\iota_{(g_j)} \tag{2.3}$$

2.4 Orbifold Cup Product

First, there is a natural map $I : X_{(g)} \longrightarrow X_{(g^{-1})}$ defined by $(p, (g)_p) \longmapsto (p, (g^{-1})_p)$.

Definition 2.4.1. Let $n = dim_{\mathbf{C}}(X)$. For any integer $0 \leq n \leq 2n$, the pairing

$$<,>_{orb}: H^d_{orb}(X) \times H^{2n-d}_{orb}(X) \longrightarrow \mathbf{Q}$$

is defined by taking the direct sum of

$$<,>^{(g)}_{orb}: H^{d-2\iota(g)}(X_{(g)}; \mathbf{Q}) \times H^{2n-d-2\iota(g^{-1})}(X_{(g^{-1})}; \mathbf{Q}) \longrightarrow \mathbf{Q}$$

where

$$< \alpha, \beta >^{(g)}_{orb} = \int^{orb}_{X_{(g)}} \alpha \wedge I^*(\beta)$$

for $\alpha \in H^{d-2\iota(g)}(X_{(g)}; \mathbf{Q})$, and $\beta \in H^{2n-d-2\iota(g^{-1})}(X_{(g^{-1})}; \mathbf{Q})$.

Choose an orbifold connection A on $E_{(\mathbf{g})}$. Let $e_A(E_{(\mathbf{g})})$ be the Euler form computed from the connection A by Chen-Weil theory. Let $\eta_j \in H^{d_j}(X_{(g_j)}; \mathbf{Q})$, for $j = 1, 2, 3$. Define maps $e_j : X_{(\mathbf{g})} \longrightarrow X_{(g_j)}$ by $(p, (\mathbf{g})_p) \longmapsto (p, (g_j)_p)$.

Definition 2.4.2. Define the 3-point function to be

$$< \eta_1, \eta_2, \eta_3 >_{orb} := \int^{orb}_{X_{(\mathbf{g})}} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_A(E_{(\mathbf{g})}) \tag{2.4}$$

Note that the above integral does not depend on the choice of A. As in the definition 2.4.1, we extend the 3-point function to $H^*_{orb}(X)$ by linearity. We define the orbifold cup product by the relation

$$< \eta_1 \cup_{orb} \eta_2, \eta_3 >_{orb} := < \eta_1, \eta_2, \eta_3 >_{orb} \tag{2.5}$$

Again we extend $\cup_{orb}$ to $H^*_{orb}(X)$ via linearity. Note that if $(\mathbf{g}) = (1, 1, 1)$, then $\eta_1 \cup_{orb} \eta_2$ is just the ordinary cup product $\eta_1 \cup \eta_2$ in $H^*(X)$.

2.5 Simplicial Toric Varieties as Orbifolds

A toric variety is a normal variety with an action of an algebraic torus which admits an open dense orbit homeomorphic to the torus. Every toric variety is described by a set of combinatoric data, called a fan Ξ in a lattice N, [13], [20]. Ξ is simplicial if every cone σ in Ξ is generated by a subset of a basis of $\mathbf{R}^n = N \otimes \mathbf{R}$.

We now describe the orbifold structure of simplicial toric varieties, see Poddar [20]. Let Ξ be any simplicial fan in a n dimensional lattice N. X_Ξ be the corresponding toric variety. For a cone $\tau \in \Xi$, denote the set of its primitive 1-dimensional generators by $\tau[1]$, the corresponding affine open subset of X_Ξ by U_τ, and the corresponding torus orbit by O_τ. We write $\nu \leq \tau$ if the cone ν is a face of the cone τ, and $\nu < \tau$ if it is a proper subface. $U_\tau = \sqcup_{\nu \leq \tau} O_\nu$. Let $M = Hom(N, \mathbf{Z})$ be the dual lattice of N with dual pair $<,>$. For any cone $\tau \in \Xi$, denote its dual cone in $M \otimes \mathbf{R}$ by $\check{\tau}$. Let $S_\tau = \check{\tau} \cap M$. $\mathbf{C}(\mathbf{S}_\tau)$ is the $\mathbf{C}$-algebra with generators χ^m for each $m \in S_\tau$ and relation $\chi^m \chi^{m'} = \chi^{m+m'}$. $U_\tau = Spec(\mathbf{C}[S_\tau])$.

Then the orbifold structure of the toric variety X_Ξ can be described as follows. Let σ be any n dimensional cone of Ξ. Let $v_1, \cdots, v_n$ be the primitive 1 dimension generstors of σ. These are linearly independent in $N_\mathbf{R} = N \otimes \mathbf{R}$. Let N_σ be the sublattice of N generated by $v_1, \cdots, v_n$. And let $G_\sigma = N/N_\sigma$ be the quotient group, then G_σ is finite and abelian.

Let σ' be the cone σ regarded in N_σ. Let $\check{\sigma}'$ be the dual cone of σ' in M_σ, the dual lattice of N_σ. $U_{\sigma'} = spec(\mathbf{C}[\check{\sigma}' \cap M_\sigma])$. Note that σ' is a smooth cone in N_σ. So $U_{\sigma'} \cong \mathbf{C}^n$.

Now there is a canonical dual pairing $M_\sigma/M \times N/N_\sigma \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow \mathbf{C}^*$, the first map by the pairing $<,>$ and the second by $q \longmapsto exp(2\pi i q)$. Now G_σ acts on $\mathbf{C}[M_\sigma]$, the group ring of M_σ, by: $v(\chi^u) = exp(2\pi i < u, v >)\chi^u$, for $v \in N$ and $u \in M_\sigma$. Note that

$$(\mathbf{C}[M_\sigma])^{G_\sigma} = \mathbf{C}[M] \tag{2.6}$$

Thus G_σ acts on $U_{\sigma'}$. Let π_σ be the quotient map. Then $U_\sigma = U_{\sigma'}/G_\sigma$. So U_σ is uniformized by $(U_{\sigma'}, G_\sigma, \pi_\sigma)$. For any $\tau < \sigma$, the orbifold structure on U_τ is the same as the one induced from the uniformizing system on U_σ. Then by the description of the toric gluing it is clear that $\{(U_{\sigma'}, G_\sigma, \pi_\sigma) : \sigma \in \Xi[n]\}$ defines a reduced orbifold structure on X_Ξ. We give a more explicit verification of this fact below.

Let B be the nonsingular matrix with generators $v_1, \cdots, v_n$ of σ as rows. Then $\check{\sigma}'$ is generated in M_σ by the column vectors $v^1, \cdots, v^n$ of the matrix B^{-1}. So $\chi^{v^1}, \cdots, \chi^{v^n}$ are the coordinates of $U_{\sigma'}$. For any $k = (k_1, \cdots, k_n) \in N$, the corresponding coset $[k] \in G_\sigma$ acts on $U_{\sigma'}$ in these coordinates as a diagonal matrix: $diag(exp(2\pi i c_1), \cdots, exp(2\pi i c_n))$, where $c_i =< k, v^i >$. Such a matrix is uniquely represented by an n-tuple $a = (a_1, \cdots, a_n)$ where $a_i \in [0, 1)$ and $c_i = a_i + b_i, b_i \in \mathbf{Z}$. In matrix notation, $kB^{-1} = a + b \cong k = aB + bB$. We denote the integral vector aB in N by k_a and the diagonal matrix corresponding to a by g_a. $k_a \longleftrightarrow g_a$ gives a one to one correspondence between the elements of G_σ and the integral vector in N that are linear combinations of the generators of σ with coefficient in $[0, 1)$.

Now let us examine the orbifold chart induced by $(U_{\sigma'}, G_\sigma, \pi_\sigma)$ at any point $p \in U_\sigma$. By the orbit decomposition, there is a unique $\tau \in \Xi$ such that $p \in O_\tau$. We assume τ is generated by $v_1, \cdots, v_j, j \leq n$. Then any preimage of p with respect to π_σ has coordinates $\chi^{v^i} = 0$ iff $i \leq j$. Let $z = (0, \ldots, 0, z_{j+1}, \ldots, z_n)$ be one such preimage. Let $G_\tau := \{g_a \in G_\sigma : a_i = 0 \; if \; j+1 \leq i \leq n\}$. We can find a small neighborhood $W \subset (\mathbf{C}^*)^{n-j}$

of $(z_{j+1}, \ldots, z_n)$ such that the inclusions $\mathbf{C}^j \times W \hookrightarrow U_{\sigma'}$ and $G_\tau \hookrightarrow G_\sigma$ induces an injection of uniformizing systems $(\mathbf{C}^j \times W, G_\tau, \pi) \hookrightarrow (U_{\sigma'}, G_\sigma, \pi_\sigma)$ on some small open neighborhood U_p of p. So we have $G_p = G_\tau$ and an orbifold chart $(\mathbf{C}^j \times W, G_\tau, \pi)$. Note that G_τ can be constructed from the set $\{k_a = \Sigma_{i=1}^j a_i v_i : k_a \in N, a_i \in [0,1)\}$ which is completely determined by τ and hence is independent of σ.

3 The Weighted Projective Spaces

3.1 The Definition and the Orbifold Structure of the Weighted Projective Space

Throughout this paper, $a(\mathbf{p})$ for $a \in \mathbf{C}$ and $\mathbf{p} = (p_1, \cdots, p_n) \in \mathbf{Z}^n$ will denote the diagonal matrix:

$$Diag(a^{p_1}, \cdots, a^{p_n})$$

with diagonal entries $a^{p_i}, i = 1, \cdots, n$. Moreover, for an integer q, μ_q will denote the group $\mathbf{Z}/q\mathbf{Z}$.

Definition 3.1.1. ([14]) Let $Q = (q_0, \cdots, q_n)$ be a $(n+1)$-tuple of positive integers. The weighted projective space of type Q, $\mathbf{P}^n(Q) = \mathbf{P}^n_{q_0, \cdots, q_n}$ is defined by

$$\mathbf{P}^n_{q_0, \cdots, q_n} = \{z \in (\mathbf{C}^{n+1})^* | z \sim \lambda(\mathbf{q}) \cdot z, \lambda \in \mathbf{C}^*\}$$

where $\lambda(\mathbf{q}) = Diag(\lambda^{q_0}, \cdots, \lambda^{q_n})$.

Remark 3.1.2. (1)The above $\mathbf{C}^*$-action is free iff $q_i = 1$ for every $i = 0, \cdots, n$; (2) If $gcd(q_0, \cdots, q_n) = d \neq 1$, then $\mathbf{P}^n_{q_0, \cdots, q_n}$ is homeomorphic to $\mathbf{P}^n_{q_0/d, \cdots, q_n/d}$(by identification of λ^d with λ).

Weighted projective spaces are, in general, orbifolds where the singularities have cyclic structure groups acting diagonally. Moreover, if all the $q_i's$ are mutually prime, all these orbifold singularities are isolated. In fact, as is usually done for complex projective spaces, we can consider the sets

$$U_i = \{[z]_Q \in \mathbf{P}^n_{q_0, \cdots, q_n} : z_i \neq 0\} \subset \mathbf{P}^n_{q_0, \cdots, q_n}$$

and the bijective maps ϕ_i from U_i to $\mathbf{C}^n/\mu_{q_i}(Q_i)$ given by

$$\phi_i([z]_Q) = \left(\frac{z_0}{(z_i)^{q_0/q_i}}, \cdots, \frac{\hat{z_i}}{z_i}, \cdots, \frac{z_n}{(z_i)^{q_n/q_i}}\right)_{q_i}$$

where $(z_i)^{1/q_i}$ is a q_i-root of z_i and $(\cdot)_{q_i}$ is a μ_i-conjugacy class in $\mathbf{C}^n/\mu_{q_i}(Q_i)$ with μ_{q_i} acting on $\mathbf{C}^n$ by

$$\xi \cdot z = \xi(Q_i)z, \xi \in \mu_{q_i}$$

Here $Q_i = (q_0, \cdots, \hat{q_i}, \cdots, q_n)$. Then on $\phi_i(U_j \cap U_i) \subset \mathbf{C}^n/\mu_{q_i}(Q_i)$,

$$\phi_j \circ \phi_i^{-1}((z_1, \cdots, z_n)_{q_i}) = \left(\frac{z_0}{(z_j)^{q_0/q_j}}, \cdots, \frac{\hat{z_j}}{z_j}, \cdots, \frac{1}{(z_j)^{q_i/q_j}}, \cdots, \frac{z_n}{(z_j)^{q_n/q_j}}\right)_{q_j}$$

3.2 Toric Structure of the Weighted Projective Spaces

Given $Q = (q_0, \cdots ,$
$q_n) \in \mathbf{Z}^{n+1}$, let $d_{i_1,\cdots,i_s} = gcd(q_{i_1}, \cdots, q_{i_s})$ and $d_j = gcd(q_0, \cdots, \hat{q}_j, \cdots, q_n)$ $(i_1, \cdots, i_s, j \in \{1, \cdots, n\})$. Define a grading of $\mathbf{C}[X_0, \cdots, X_n]$ by $degX_i = q_i$. We denote this ring by $S(Q)$. Then $\mathbf{P}^n_{q_0,\cdots,q_n} = projS(Q)$ is the weighted projective space of type Q. $\mathbf{P}^n_{q_0,\cdots,q_n}$ is covered by the affine open sets $D_+(X_i) := specS(Q)_{X_i}, (i = 0, ..., n)$. The monic monomials of $S(Q)_{X_i}$ are of type $X_i^{-l}\Pi_{j\neq i}X_j^{\lambda_j}$, where $lq_i = \Sigma_{j\neq i}\lambda_j q_j$ and l, λ_j are non-negative integers. So each such monomial is uniquely determined by the n-tuple $(\lambda_0, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n)$ of its non-negative exponents. The exponents occurring are just the points lying in the intersection of the cone $e :\doteq pos\{e_1, \cdots, e_n\}$ and the lattice $N_{Q,q_i} \subseteq \mathbf{Z}^n$ that is defined as follows.

Consider $Q_i = (q_0, \cdots, \hat{q}_i, \cdots, q_n)$ as an element of $Hom_{\mathbf{Z}}(\mathbf{Z}^n, \mathbf{Z})$ by setting:

$$Q_i(a_1, \cdots, a_n) := q_0 a_1 + \cdots, +q_n a_n$$

$\mathbf{Z}^n$ being equipped with its canonical basis. Let $\pi_i : \mathbf{Z} \longrightarrow \mathbf{Z}_{q_i}$ denote the canonical projection. Then

$$N_{Q,q_i} := Ker(\mathbf{Z}^n \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}_{q_i})$$

is a sublattice of $\mathbf{Z}^n$. Denote by $M_{Q,i}$ the dual lattice. We have an isomorphism of semigroup rings

$$S(Q)_{(X_i)} \cong \mathbf{C}[e \cap N_{Q,q_i}]$$

revealing $D_+(X_i)$ to be the affine toric variety associated with $\check{e}$ with respect to M_{Q,q_i}.

Proposition 3.2.1. *([5]) Let $C_i = (c_1^i, \cdots, c_n^i)$ be a basis of N_{Q,q_i} and denote by $r_1^i, \cdots, r_n^i$ the row vectors of C_i. Let $\sigma_i := pos\{r_1^i, \cdots, r_n^i\}$. Then there is an isomorphism of semigroups*

$$\check{\sigma}_i \cap \mathbf{Z}^n \simeq e \cap N_{Q,q_i}.$$

Proposition 3.2.2. *([5]) With the natation introduced above, the matrix*

$$C_0 = \begin{pmatrix} \frac{q_0}{d_{01}} & c_{12}^0 & \cdots & c_{1n}^0 \\ 0 & \frac{d_{01}}{d_{012}} & \cdots & c_{2n}^0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{d_n}{d_{0\cdots n}} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = (c_{ij}^0)$$

is a basis of N_{Q,q_0}, where $c_{ij}^0 \in \mathbf{Z}_{\geq 0}$ are determined as follows. For fixed $j \in \{2, ..., n\}$ construct c_{ij}^0 successively for $i = j-1, ..., 1$ by requiring $c_{ij}^0 \in \mathbf{Z}_{\geq 0}$ to be minimal with the property that

$$c_{ij}^0 q_i + \sum_{\nu=i+1}^{j} c_{\nu j}^0 q_\nu \in gcd(q_0, ..., q_{i-1})\mathbf{Z} = d_{0\cdots i-1}\mathbf{Z}.$$

Proposition 3.2.3. *([5]) With the notations introduced above, let $v_0 := -\sum_{i=1}^n \frac{q_i}{q_0} v_i$, $\rho_i = \mathbf{R}_{\geq 0} v_i$. Then the complete fan Ξ determined by $\Xi[1] = \{\rho_0, \cdots, \rho_n\}$ is the fan of the weighted projective space $\mathbf{P}^n_{q_0,\cdots,q_n}$.*

Remark 3.2.4. (1) The weighted projective space $\mathbf{P}^n_{q_0,\cdots,q_n}$ can also be constructed as follows. Given a fan $\Xi = \{u_0, \cdots, u_n\}$ so that $q_0u_0 + q_1u_1 + \cdots + q_nu_n = 0$, then the toric variety X_Ξ is the weighted projective space $\mathbf{P}^n_{q_0,\cdots,q_n}$. From above we can see that $\{v_0, \cdots, v_n\}$ satisfies the condition $q_0v_0 + q_1v_1 + \cdots + q_nv_n = 0$. In fact, the proposition 3.2.2 and 3.2.3 gave a method to compute the fan of weighted projective space of type Q. (2) If $gcd(q_0, \cdots, q_n) = d \neq 1$, from the construction of the matrix C_0, we can see that $\mathbf{P}^n_{q_0,\cdots,q_n}$ and $\mathbf{P}^n_{q_0/d,\cdots,q_n/d}$ have the same fans, so they are homeomorphic.

3.3 Homogeneous Coordinate Representations of Weighted Projective Spaces Induced from Toric Varieties

In this section we use the theorem of David Cox (see [9]) to represent the weighted projective space $\mathbf{P}^n_{q_0,\cdots,q_n}$ as the geometric quotient $(\mathbf{C}^{\mathbf{n+1}})^*/\mathbf{C}^*$. From the theorem of David Cox in [8], for a fan Ξ, the toric variety X_Ξ is a geometric quotient $(\mathbf{C})^r\backslash Z/G$ iff Ξ is simplicial, where r is the number of the 1-primitive generators,Z is a subvariety of $\mathbf{C}^r$, G is some subgroup of $(\mathbf{C}^*)^r$. To give this representation for the weighted projective space, we must compute the space Z and the group G. Let $\Xi = \{v_0, \cdots, v_n\}$ be the fan of weighted projective space $\mathbf{P}^n_{q_0,\cdots,q_n}$. Then there are $n+1$ 1-dimensional primitive generators, which give variables $x_0, \cdots, x_n$. Furthermore, the maximal cones of the fan are generated by the n-element subsets of $\{v_0, \cdots, v_n\}$. It follows from [9] that

$$Z = V(x_0, \cdots, x_n) = \{(0, \cdots, 0)\} \subset \mathbf{C}^{n+1}.$$

Now we describe the group G. From [9],

$$G = \left\{(\mu_0, \cdots, \mu_n) \in (\mathbf{C}^*)^{n+1} | \Pi_{i=1}^{n+1} \mu_i^{<m,v_i>} = 1, \text{for all } m \in \mathbf{Z}^n\right\}$$

However it suffices to let m be the standard basis elements $e_1, \cdots, e_n$. Thus $(\mu_0, \cdots, \mu_n) \in G$ iff

$$\Pi_{i=0}^n \mu_i^{<e_1,r_i>} = \Pi_{i=0}^n \mu_i^{<e_2,r_i>} = \cdots = \Pi_{i=0}^n \mu_i^{<e_n,r_i>} = 1 \qquad (3.1)$$

From proposition 3.2.3, we have the vectors $v_1, \cdots, v_n$, and $v_0 = -\Sigma_{i=1}^n \frac{q_i}{q_0} r_i$, so we have:

$$\begin{aligned} \mu_0^{-\frac{q_1}{d_{01}}} \mu_1^{\frac{q_0}{d_{01}}} &= \mu_0^{(-\frac{q_1}{q_0}c^0_{12} - \frac{q_2 d_{01}}{q_0 d_{012}})} \mu_1^{c^0_{12}} \mu_2^{\frac{d_{01}}{d_{012}}} = \cdots = \\ &= \mu_0^{(-\frac{q_1}{q_0}c^0_{1n} - \frac{q_2}{q_0}c^0_{2n} - \cdots - \frac{q_n d_n}{q_0 d_{0\cdots n}})} \mu_1^{c^0_{1n}} \cdots \mu_{n-1}^{c^0_{n-1,n}} \mu_n^{\frac{d_n}{d_{0\cdots n}}} = 1 \end{aligned}$$

So

$$\begin{cases} \mu_0^{-\frac{q_1}{d_{01}}} = \mu_1^{\frac{q_0}{d_{01}}}, \\ \mu_0^{(\frac{q_1}{q_0}c^0_{12} + \frac{q_2 d_{01}}{q_0 d_{012}})} = \mu_1^{c^0_{12}} \mu_2^{\frac{d_{01}}{d_{012}}}; \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_0^{(\frac{q_1}{q_0}c^0_{1n} + \frac{q_2}{q_0}c^0_{2n} + \cdots + \frac{q_n d_n}{q_0 d_{0\cdots n}})} = \mu_1^{c^0_{1n}} \cdots \mu_{n-1}^{c^0_{n-1,n}} \mu_n^{\frac{d_n}{d_{0\cdots n}}} \end{cases}$$