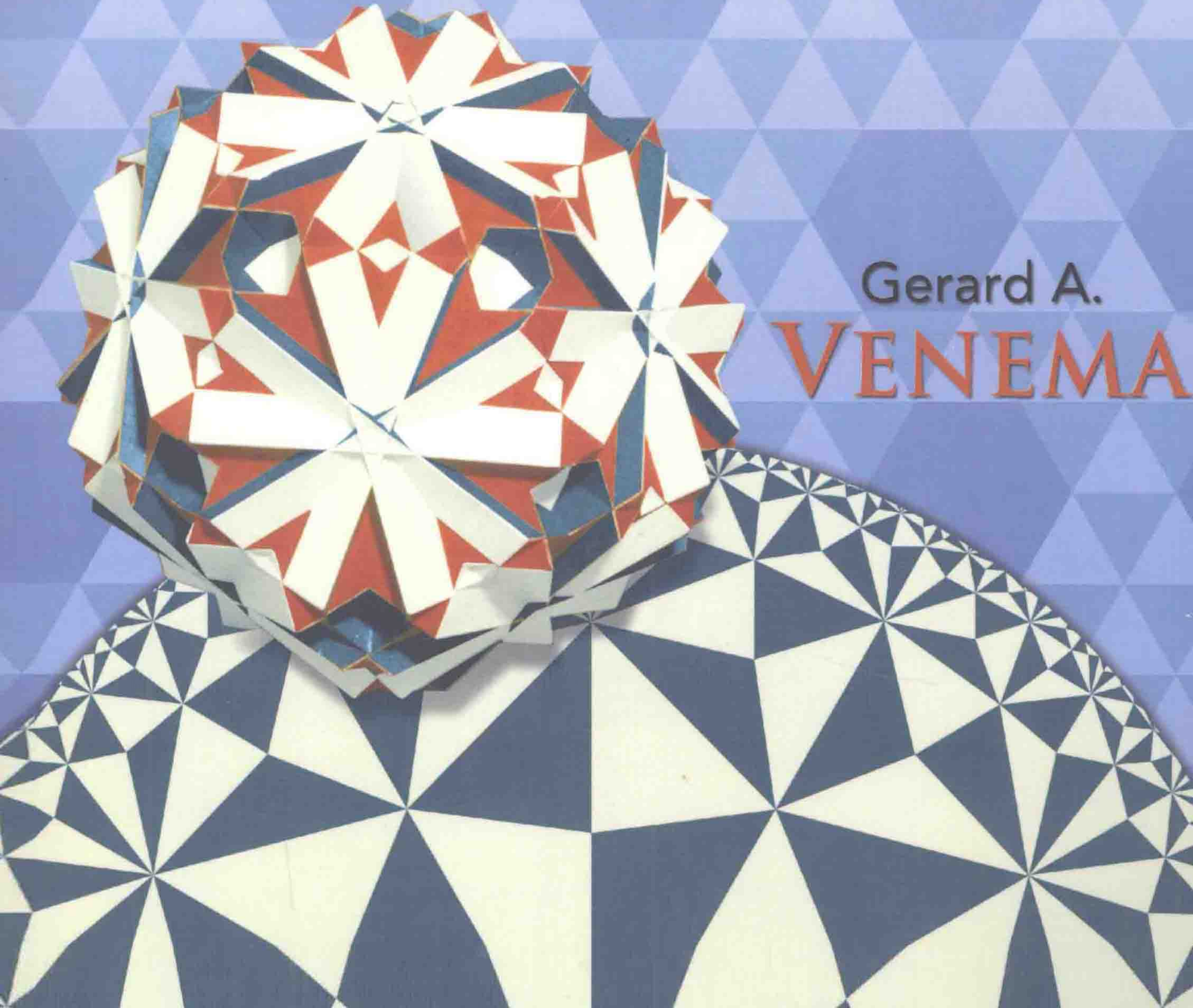
Foundations of GEOMETRY

Second Edition



Gerard A. VENEMA

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To my family:

Patricia, Sara, Emily, Daniel, David, Christian, and Ian.

Preface

This is a textbook for an undergraduate course in axiomatic geometry. The text is targeted at mathematics students who have completed the calculus sequence and perhaps a first course in linear algebra, but who have not yet encountered such upper-level mathematics courses as real analysis and abstract algebra. A course based on this book will enrich the education of all mathematics majors and will ease their transition into more advanced mathematics courses. The book also includes emphases that make it especially appropriate as the textbook for a geometry course taken by future high school mathematics teachers.

WHAT'S NEW IN THE SECOND EDITION?

For the benefit of those who have used the first edition of this book, here is a quick summary of what has changed in the second edition.

- The first part of the text has been extensively reorganized and streamlined to make it possible to reach the chapter on neutral geometry more quickly.
- More exercises have been added throughout.
- The technology sections have been rewritten to facilitate the use of GeoGebra.
- The review of proof writing has been incorporated into the chapter on axiomatic systems.
- Many of the theorems in the early chapters are first stated in an informal, intuitive way and then are formally restated in if-then form.
- The organization of the chapter on neutral geometry has been tightened up.
- The coverage of asymptotically parallel lines in hyperbolic geometry was expanded and there is now a complete proof of the classification of parallels.
- A section on similarity transformations was added.
- The material on set theory and the real numbers was moved to an appendix.
- The description of various axiom systems for elementary geometry that came at the beginning of the chapter on Axioms for Plane Geometry has been expanded and moved to a new appendix where it can be covered at any time during the course.

THE FOUNDATIONS OF GEOMETRY

The principal goal of the text is to study the foundations of geometry. That means returning to the beginnings of geometry, exposing exactly what is assumed there, and building the entire subject on those foundations. Such careful attention to the foundations has a long tradition in geometry, going back more than two thousand years to Euclid and the ancient Greeks. Over the years since Euclid wrote his famous *Elements*, there have been profound changes in the way in which the foundations have been understood. Most of those changes have been byproducts of efforts to understand the true place of Euclid's parallel postulate in the foundations, so the parallel postulate is one of the primary emphases of this book.

ORGANIZATION OF THE BOOK

The book begins with a brief look at Euclid's *Elements*, and Euclid's method of organization is used as motivation for the concept of an axiomatic system. A system of axioms for geometry is then carefully laid out. The axioms used here are based on the real numbers, in the spirit of Birkhoff, and their statements have been kept as close to those in contemporary high school textbooks as is possible.

After the axioms have been stated and certain foundational issues faced, neutral geometry, in which no parallel postulate is assumed, is extensively explored. Next both Euclidean and hyperbolic geometries are investigated from an axiomatic point of view. In order to get as quickly as possible to some of the interesting results of non-Euclidean geometry, the first part of the book focuses exclusively on results regarding lines, parallelism, and triangles. Only after those topics have been treated separately in neutral, Euclidean, and hyperbolic geometries are results on area, circles, and construction introduced. While the treatment of these subjects does not exactly follow Euclid, it roughly parallels Euclid in the sense that Euclid collected most of his propositions about area in Book II and most of his propositions about circles in Books III and IV. The three chapters covering area, circles, and construction complete the coverage of the major theorems of Books I through VI of the *Elements*.

The more modern notion of a transformation is introduced next and some of the standard results regarding transformations of the plane are explored. A complete proof of the classification of the rigid motions of both the Euclidean and hyperbolic planes is included. There is a discussion of how the foundations of geometry can be reorganized to reflect the transformational point of view (as is common practice in contemporary high school geometry textbooks). Specifically, it is possible to replace the Side-Angle-Side Postulate with a postulate that asserts the existence of certain reflections.

The standard models for hyperbolic geometry are carefully constructed and the results of the chapter on transformations are used to verify their properties. The chapter on models can be relatively short because all the hard technical work involved in the constructions is done in the preceding chapter. The final chapter includes a study of some of the polygonal models that have recently been developed to help students understand what it means to say that hyperbolic space is negatively curved. The book ends with a discussion of the practical significance of non-Euclidean geometry and a brief look at the geometry of the real world.

NATIONAL STANDARDS

A significant portion of the audience for a course based on this text consists of future high school geometry teachers. In order to meet the needs of that group of mathematics majors, current national standards regarding the mathematical education of teachers and the content of the high school geometry curriculum were consulted in the design of the text. An important secondary goal of the text is to implement the recommendations in two recent sets of standards: the 2001 report on *The Mathematical Education of Teachers* [6] and the 2010 *Common Core State Standards for Mathematics* [7].

The recommendations of *The Mathematical Education of Teachers* (MET) are based on the "Principles and Standards for School Mathematics" of the National Council of Teachers of Mathematics [36]. The principal recommendation of MET is that "Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach" [6, Part I, page 7]. This text is designed to do precisely that in the area of geometry. A second basic recommendation in MET is that courses for prospective mathematics teachers should make explicit connections with high school mathematics. Again, this book attempts to implement that recommendation in geometry. The goal is to follow the basic recommendations of MET, not necessarily to cover every geometric topic that future teachers need to see; some geometric topics will be included in other courses, such as linear algebra.

An example of the way in which connections with high school geometry have influenced the design of the text is the choice of the axioms that are used as the starting point. The axioms on which the development of the geometry in the text is based are almost exactly those that are used in high school textbooks. While most high school textbooks still include an axiomatic treatment of geometry, there is no standard set of axioms that is common to all high school geometry courses. Therefore, various axiom systems are considered in an appendix and the merits and advantages of each are discussed. The axioms on which this text is ultimately based are as close as possible to those in contemporary high school textbooks. One of the main goals of the text is to help preservice teachers understand the logical foundations of the geometry course they will teach and that can best be accomplished in the context of axioms that are like the ones they will encounter later in the classroom. There are many other connections with high school geometry that are brought in as the text progresses.

The *Common Core State Standards for Mathematics* (CCSSM) specify what should be included in the high school geometry curriculum. This book attempts to give future teachers a grounding in the themes and perspectives described there. In particular, there is an emphasis on Euclidean geometry and the parallel postulate. The transformational approach to congruence and similarity, the approach that is promoted by CCSSM, is studied in Chapter 10 and is related to other, more traditional, ways of interpreting congruence. All of the specific topics listed in CCSSM are covered in the text. Finally, CCSSM mentions that "...in college some students will develop Euclidean and other geometries carefully from a small set of axioms." A course based on this textbook is exactly the kind of course envisioned in that remark.

Some of the newer high school mathematics curricula present mathematics in an integrated way that emphasizes connections between the branches of mathematics. There is no separate course in geometry, but rather a geometry thread is woven into all the high school mathematics courses. In order to teach such a course well, the teacher herself needs to have an understanding of the structure of geometry as a coherent subject. This book is intended to provide such an understanding.

One of the recurring themes in MET is the recommendation that prospective teachers must acquire an understanding of high school mathematics that goes well beyond that of a typical high school graduate. One way in which such understanding of geometry is often measured is in terms of the van Hiele model of geometric thought. This model is described in Appendix D. The goal of most high school courses is to develop student thinking to Level 3. A goal of this text is to bring students to Level 4 (or to Level 5, depending on whether the first level is numbered 0 or 1). It is recognized, however, that not all students entering the course are already at Level 3 and so the early part of the text is designed to ensure that students are brought to that level first.

PROOFS

A third goal of the text is to teach the art of writing proofs. There is a growing recognition of the need for a course in which mathematics students learn how to write good proofs. Such a course should serve as a bridge between the lower-level mathematics courses, which are largely technique oriented, and the upper-level courses, which tend to be much more conceptual. This book uses geometry as the vehicle for helping students to write and appreciate proofs. The ability to write proofs is a skill that can only be acquired by actually practicing it, so most of the material on writing proofs is integrated into the text and the attention to proof permeates the entire text. This means that the book can also be used in classes where the students already have experience writing proofs; despite the emphasis on writing proofs, the book is still primarily a geometry text.

Having the geometry course serve as the introduction to proof represents a return to tradition in that the course in Euclidean geometry has for thousands of years been seen as the standard introduction to logic, rigor, and proof in mathematics. Using the geometry course this way makes historical sense because the axiomatic method was first introduced in geometry and geometry remains the branch of mathematics in which that method has had its greatest success. While proof and logical deduction are still emphasized in the standards for high school mathematics, most high school students no longer take a full-year course devoted exclusively to geometry with a sustained emphasis on proof. This makes it more important than ever that we teach a good college-level geometry course to all mathematics students. By doing so we can return geometry to its place as the subject in which students first learn to appreciate the importance of clearly spelling out assumptions and deducing results from those assumptions via careful logical reasoning.

The emphasis on proof makes the course a do-it-yourself course in that the reader will be asked to supply proofs for many of the key theorems. Students who diligently work the exercises come away from the course with a sense that they have an unusually deep understanding of the material. In this way the student should not only learn the mechanics of good proof writing style but should also come to more fully appreciate the important role proof plays in an understanding of mathematics.

HISTORICAL AND PHILOSOPHICAL PERSPECTIVE

A final goal of the text is to present a historical perspective on geometry. Geometry is a dynamic subject that has changed over time. It is a part of human culture that was created and developed by people who were very much products of their time and place. The foundations of geometry have been challenged and reformulated over the years, and beliefs about the relationship between geometry and the real world have been challenged as well.

The material in the book is presented in a way that is sensitive to such historical and philosophical issues. This does not mean that the material is presented in a strictly historical order or that there are lengthy historical discussions but rather that geometry is presented in such a way that the reader can understand and appreciate the historical development of the subject and so that it would be natural to investigate the history of the subject while learning it. Many chapters include suggested readings on the history of geometry that can be used to enrich the text.

Throughout the book there are references to philosophical issues that arise in geometry. For example, one question that naturally occurs to anyone studying non-Euclidean geometry is this: What is the connection between the abstract entities that are studied in a course on the foundations of geometry and properties of physical space? The book does not present dogmatic answers to such questions, but instead simply raises them in an effort to promote student thinking. The hope is that this will serve to counter the common perception that mathematics is a subject in which every question has a single correct answer and in which there is no room for creative ideas or opinions.

WHY AXIOMATIC GEOMETRY?

One question I am often asked is: Why study axiomatic geometry? Why take such an old-fashioned approach to geometry when there are so many beautiful and exciting modern topics that could be included in the course? The main answer I give is that proof and the axiomatic method remain hallmarks of mathematics. In order to be well educated in mathematics, students should see a full axiomatic development of a complete branch of mathematics. They need to know about the historical importance of the axiomatic approach in geometry and they need to be aware of the profound changes in our understanding of the relationship between mathematics and the real world that grew out of attempts to understand the place of the parallel postulate.

For many reasons, a course in axiomatic (Euclidean) geometry is a natural setting in which students can learn to write and appreciate proofs.

- The objects studied are natural and familiar.
- The definitions are uncomplicated, requiring few quantifiers.
- All necessary assumptions can be completely described.
- The proofs are relatively straightforward and the ideas can be understood visually.
- The proofs usually require a nontrivial idea, so students appreciate the need for a proof.

The proofs in the early books of Euclid's *Elements* are beautiful, yet simple, containing just the right amount of detail. Studying the proofs of Euclid remains one of the best introductions to proofs and the geometry course can serve as a bridge to higher mathematics for all mathematics majors. Another justification for a college course in axiomatic geometry is that most students no longer have the experience of studying axiomatic geometry in high school.

WHY EUCLIDEAN GEOMETRY?

Another question I am frequently asked is: Why include such an extensive treatment of Euclidean geometry in a college course? Don't students learn enough about that subject in high school? The answer, sadly, is that students are not learning enough about Euclidean geometry in high school. Most high school geometry courses no longer include a study of the axioms and some do not emphasize proof at all. In many cases there is not even a separate course in geometry, but geometry is one of several threads that are woven together in the high school mathematics curriculum. Rather than lamenting these developments, I think college and university mathematics departments should embrace the opportunity to teach a substantial geometry course at the college level. This can restore Euclidean geometry to its traditional role as the course in which mathematics students have their first experience with careful logical thinking and the complete development of a comprehensive, coherent mathematical subject.

TECHNOLOGY

In recent years powerful computer software has been developed that can be used to explore geometry. The study of geometry from this book can be greatly enhanced by such dynamic software and the reader is encouraged to find appropriate ways in which to incorporate this technology into the geometry course. While software can enrich the experience of learning geometry from this book, its use is not required. The book can be read and studied quite profitably without it.

The author recommends the use of the dynamic mathematics software program GeoGebra. GeoGebra is free software that is intended to be used for teaching and learning mathematics. It may be downloaded from the website www.geogebra.org. The software has many great features that make it ideal for use in the geometry classroom, but the main advantage it has over commercial geometry software is the fact that it is free and runs under any of the standard computer operating systems. This means that students can load the program on their own computers and will always have access to it.

Any of the several commercially available pieces of dynamic geometry software will also serve the purpose. *Geometer's Sketchpad*™ (Key Curriculum Press) is widely used and readily available. *Cabri Geometry*™ (Texas Instruments) is less commonly used in college-level courses, but it is also completely adequate. It has some predefined tools, such as an inversion tool and a test for collinearity, that are not included in Sketchpad. *Cinderella*™ (Springer-Verlag) is Java-based software and is also very good. It has the advantage that it allows diagrams to be drawn in all three two-dimensional geometries: Euclidean, hyperbolic, and spherical. Another advantage is that it allows diagrams to be easily exported as Java applets. A program called NonEuclid is freely available on the internet and it can be used to enhance the non-Euclidean geometry in the course. New software is being produced all the time, so you may find that other products are available to you.

This is a course in the foundations of axiomatic geometry, and software will necessarily play a more limited role in such a course than it might in other kinds of geometry courses. Nonetheless, there is an appropriate role for software in a course such as this and the author hopes that the book will demonstrate that. There is no reason for those who love the proofs of Euclid to resist the use of technology. After all, Euclid himself made use of the limited technology that was available to him, namely the compass and straightedge. In the same way we can make good use of modern technology in our study of geometry. It is especially important that future high school teachers learn to understand and appreciate the *appropriate* use of technology.

In the first part of the text (Chapters 2 through 4), the objective is to carefully expose all the assumptions that form the foundations of geometry and to understand for ourselves how the basic results of geometry are built on those foundations. For most users the software is a black box in the sense that we either don't know what assumptions are built into it or we have only the authors' description of what went into the software. As a result, software is of limited use in this part of the course and it will not be mentioned explicitly in the first four chapters of the book. But you should be using it to draw diagrams and to experiment with what happens when you vary the data in the theorems. During that phase of the course the main function of the software is to illustrate one possible interpretation of the relationships being studied.

It is in the second half of the course that the software comes into its own. Computer software is ideal for experimenting, exploring, and discovering new relationships. In order to illustrate that, several of the later chapters include sections in which the software is used to explore ideas that go beyond those that are presented in detail and to discover new relationships. In particular, there are such exploratory sections in the chapters on Euclidean geometry and circles. The entire chapter on constructions is written as an exploration with only a limited number of proofs or hints provided in the text.

The exploratory sections of the text have been expanded into a laboratory manual entitled *Exploring Advanced Euclidean Geometry with GeoGebra*.

SUPPLEMENTS

Two supplements are available: an *Instructors' Manual* and a computer laboratory manual entitled *Exploring Advanced Euclidean Geometry with GeoGebra*.

- The *Instructors' Manual* contains complete solutions to all of the exercises as well as information about how to teach from the book. Instructors may download it from
  http://www.pearsonhighered.com/irc.

- Additional materials may be downloaded, free of charge, from the author's website at
  http://calvin.edu/~venema/geometrybook.html.

DESIGNING A COURSE

A full-year course should cover essentially all the material in the text. There can be some variation based on instructor and student interest, but most or all of every chapter should be included.

An instructor teaching a one-semester or one-quarter course will be forced to pick and choose. It is important that this be done carefully so that the course reaches some of the interesting and useful material that is to be found in the second half of the book.

- Chapter 1 sets the stage for what is to come, so it should be covered in some way. But it can be discussed briefly in class and then assigned as reading.
- Chapter 2 should definitely be covered because it establishes the basic framework for the treatment of geometry that follows.
- The basic coverage of geometry begins with Chapter 3. Chapters 3 and 4 form the heart of a one-semester course. Those chapters should be included in any course taught from the book.
- At least some of Chapter 5 should also be included in any course.
- Starting with Chapter 7, the chapters are largely independent of each other and an instructor can select material from them based on the interests and needs of the class.

Several sample course outlines are included below. Many other variations are possible. It should be noted that the suggested outlines are ambitious and many instructors will choose to cover less.

A course emphasizing Euclidean Geometry

Chapter	Topic	Number of weeks
1 & 2	Preliminaries	$\leqslant 2$
3	Axioms	2
4	Neutral geometry	3
5	Euclidean geometry	2
7	Area	1–2
8	Circles	1–2
10	Transformations	2

A course emphasizing non-Euclidean Geometry

Chapter	Topic	Number of weeks
1 & 2	Preliminaries	$\leqslant 2$
3	Axioms	2
4	Neutral geometry	3
5	Euclidean geometry	1
6	Hyperbolic geometry	2
7	Area	1–2
11	Models	1–2
12	Geometry of space	1

A course for future high school teachers

Chapter	Topic	Number of weeks
1 & 2	Preliminaries	$\leqslant 2$
3	Axioms	2
4	Neutral geometry	3
5	Euclidean geometry	1
6	Hyperbolic geometry	1
7	Area	1
8	Circles	1
10	Transformations	1
11	Models	1
12	Geometry of space	1

The suggested course for future high school teachers includes just a brief introduction to each of the topics in later chapters. The idea is that the course should provide enough background so that students can study those topics in more depth later if they need to. It is hoped that this book can serve as a valuable reference for those who go on to teach geometry courses. The book could be a resource that provides information about rigorous treatments of such topics as parallel lines, area, circles, constructions, transformations, and so on, that are part of the high school curriculum.

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Gerard A. Venema
Calvin College
May 2011

Contents

APPENDICES

CHAPTER 1

Prologue: Euclid's *Elements*

Our study of geometry begins with an examination of the historical origins of the axiomatic method in geometry. While the material in this chapter is not a mathematical prerequisite for what comes later, an appreciation of the historical roots of axiomatic thinking is essential to an understanding of why the foundations of geometry are systematized as they are.

1.1 GEOMETRY BEFORE EUCLID

Geometry is an ancient subject. Its roots go back thousands of years and geometric ideas of one kind or another are found in nearly every human culture. The beauty of geometric patterns is universally appreciated and often investigated in a systematic way. The study of geometry as we know it emerged more than 4000 years ago in Mesopotamia, Egypt, India, and China.

Because the Nile River annually flooded vast areas of land and obliterated property lines, surveying and measuring were important to the ancient Egyptians. This practical interest may have motivated their study of geometry. Egyptian geometry was mostly an empirical science, consisting of many rule-of-thumb procedures that were arrived at through experimentation, observation, and trial and error. Formulas were approximate ones that appeared to work, or at least gave answers that were close enough for practical purposes. But the ancient Egyptians were also aware of more general principles, such as special cases of the Pythagorean Theorem and formulas for volumes.

The ancient Mesopotamians, or Babylonians, had an even more advanced understanding of geometry. They knew the Pythagorean Theorem long before Pythagoras. They discovered some of the area-based proofs of the theorem that will be discussed in Chapter 7, and knew a general method that generates all triples of integers that are lengths of sides of right triangles. In India, ancient texts apply the Pythagorean Theorem to geometric problems associated with the design of structures. The Pythagorean Theorem was also discovered in China at roughly the same time.

About 2500 years ago there was a profound change in the way geometry was practiced: Greek mathematicians introduced abstraction, logical deduction, and proof into geometry. They insisted that geometric results be based on logical reasoning from first principles. In theory this made the results of geometry exact, certain, and undeniable, rather than just likely or approximate. It also took geometry out of the realm of everyday experience and made it a subject that studies abstract entities. Since the purpose of this course is to study the logical foundations of geometry, it is natural that we should start with the geometry of the ancient Greeks.

The process of introducing logic into geometry apparently began with Thales of Miletus around 600 B.C. and culminated in the work of Euclid of Alexandria in approximately 300 B.C. Euclid is the most famous of the Greek geometers and his name is still universally associated with the geometry that is studied in schools today. Most of the ideas that are included in what we call "Euclidean Geometry" probably did not originate with Euclid himself; rather, Euclid's contribution was to organize and present the results of Greek geometry in a logical and coherent way. He published his results in a series of thirteen books known as his *Elements*. We begin our study of geometry by examining those *Elements* because they set the agenda for geometry for the next two millennia and more.

1.2 THE LOGICAL STRUCTURE OF EUCLID'S *ELEMENTS*

Euclid's *Elements* are organized according to strict logical rules. Euclid begins each book with a list of definitions of the technical terms he will use in that book. In Book I he next states five "postulates" and five "common notions." These are assumptions that are meant to be accepted without proof. Both the postulates and common notions are basic statements whose truth should be evident to any reasonable person. They are the starting point for what follows. Euclid recognized that it is not possible to prove everything, that he had to start somewhere, and he attempted to be clear about exactly what his assumptions were.

Most of Euclid's postulates are simple statements of intuitively obvious and undeniable facts about space. For example, Postulate I asserts that it is possible to draw a straight line through any two given points. Postulate II says that a straight line segment can be extended to a longer segment. Postulate III states that it is possible to construct a circle with any given center and radius. Traditionally these first three postulates have been associated with the tools that are used to implement them on a piece of paper. The first two postulates allow two different uses of a straight edge: A straight edge can be used to draw a line segment connecting any two points or to extend a given line segment to a longer one. The third postulate affirms that a compass can be used to construct a circle with a given center and radius. Thus the first three postulates simply permit the familiar straightedge and compass constructions of high school geometry.

The fourth postulate asserts that all right angles are congruent ("equal" in Euclid's terminology). The fifth postulate makes a more subtle and complicated assertion about two lines that are cut by a transversal. These last two postulates are the two technical facts about geometry that Euclid needs in his proofs.

The common notions are also intuitively obvious facts that Euclid plans to use in his development of geometry. The difference between the common notions and the postulates is that the common notions are not peculiar to geometry but are common to all branches of mathematics. They are everyday, common-sense assumptions. Most spell out properties of equality, at least as Euclid used the term *equal*.

The largest part of each Book of the *Elements* consists of propositions and proofs. These too are organized in a strict, logical progression. The first proposition is proved using only the postulates, Proposition 2 is proved using only the postulates and Proposition 1, and so on. Thus the entire edifice is built on just the postulates and common notions; once these are granted, everything else follows logically and inevitably from them. What is astonishing is the number and variety of propositions that can be deduced from so few assumptions.