

CALCULUS
WITH
ANALYTIC
GEOMETRY

2

JOHN M. H. OLMSTED

VOLUME II

CALCULUS
WITH ANALYTIC GEOMETRY

John M. H. Olmsted

Southern Illinois University



New York

APPLETON-CENTURY-CROFTS

DIVISION OF MEREDITH PUBLISHING COMPANY

Copyright © 1966 by

MEREDITH PUBLISHING COMPANY

All rights reserved. This book, or parts thereof, must not be used or reproduced in any manner without written permission. For information address the publisher, Appleton-Century-Crofts, Division of Meredith Publishing Company, 440 Park Avenue South, New York, New York 10016

616-1

Library of Congress Card Number: 66-11169

PRINTED IN THE UNITED STATES OF AMERICA

E 67841

CALCULUS
WITH ANALYTIC GEOMETRY

The Appleton-Century Mathematics Series

Raymond W. Brink and John M. H. Olmsted, *Editors*

A First Year of College Mathematics, 2nd ed., by Raymond W. Brink

Algebra—College Course, 2nd ed., by Raymond W. Brink

Analytic Geometry, rev. ed., by Raymond W. Brink

College Algebra, 2nd ed., by Raymond W. Brink

Essentials of Analytic Geometry by Raymond W. Brink

Intermediate Algebra, 2nd ed., by Raymond W. Brink

Plane Trigonometry, 3rd ed., by Raymond W. Brink

Spherical Trigonometry by Raymond W. Brink

An Introduction to Matrices, Vectors, and Linear Programming by Hugh G. Campbell

Modern Basic Mathematics by Hobart C. Carter

Elementary Concepts of Modern Mathematics by Flora Dinkines

Parts also available individually under the following titles:

Part I, *Elementary Theory of Sets*

Part II, *Introduction to Mathematical Logic*

Part III, *Abstract Mathematical Systems*

Introduction to the Laplace Transform by Dio L. Holl, Clair G. Maple, and Bernard Vinograd

Introductory Analysis by V. O. McBrien

College Geometry by Leslie H. Miller

Advanced Calculus by John M. H. Olmsted

Calculus with Analytic Geometry (2 volumes) by John M. H. Olmsted

Intermediate Analysis by John M. H. Olmsted

Real Variables by John M. H. Olmsted

Solid Analytic Geometry by John M. H. Olmsted

Analytic Geometry by Edwin J. Purcell

Calculus with Analytic Geometry by Edwin J. Purcell

Analytic Geometry and Calculus by Lloyd L. Smail

Calculus by Lloyd L. Smail

The Mathematics of Finance by Franklin C. Smith

TO CYNTHIA

CONTENTS

22 TECHNIQUES OF INTEGRATION

2201	Integration of the trigonometric functions.....	1
2202	Powers of sines and cosines.....	3
2203	Exercises.....	6
2204	Integration by parts.....	8
2205	Integration by means of differentiation.....	10
2206	Exercises.....	13
2207	Powers of other trigonometric functions.....	14
2208	Exercises.....	16
H2209	Repeated integration by parts.....	17
H2210	Exercises.....	19
2211	The inverse sine and inverse tangent.....	19
2212	Inverses of the other trigonometric functions.....	25
2213	Exercises.....	28
2214	Integration by trigonometric substitution.....	29
2215	Exercises.....	34
2216	Partial fractions; linear factors.....	36
2217	Quadratic factors.....	41
2218	Exercises.....	44
H2219	Proofs of partial fractions expansions.....	45
H2220	Exercises.....	48
2221	Half-angle substitutions.....	49
2222	Other substitutions.....	51
2223	Exercises.....	54
2224	Hyperbolic functions and their inverses.....	55
2225	Exercises.....	60
2226	A short table of integrals.....	62
2227	Reduction formulas; use of tables of integrals.....	64
2228	Exercises.....	66
H2229	The catenary and the tractrix.....	67
H2230	Exercises.....	70

23 IMPROPER INTEGRALS

2301	Unbounded functions on a bounded interval.....	71
2302	Exercises.....	74
2303	An unbounded interval of integration.....	76
2304	Combinations of intervals.....	77
2305	Exercises.....	78
2306	Comparison tests; dominance.....	80
2307	Big O and little o notation.....	83
2308	Exercises.....	87
H2309	The Cauchy criterion.....	88
H2310	Exercises.....	92

24 APPROXIMATION METHODS

2401	The trapezoidal rule.....	94
2402	Measure of accuracy.....	98
2403	Exercises.....	98
2404	Simpson's rule.....	99
2405	Measure of accuracy.....	102
2406	Exercises.....	103
2407	Newton's method.....	103
2408	Exercises.....	107
H2409	Proofs of measures of accuracy.....	108
H2410	Exercises.....	110

25 INFINITE SERIES

2501	Sequences.....	111
2502	Convergence and divergence.....	113
2503	Exercises.....	117
2504	Fundamental properties of sequences.....	119
2505	Exercises.....	122
2506	Infinite series.....	123
2507	Nonnegative and positive series; the integral test.....	127
2508	Exercises.....	130
H2509	The Cauchy criterion.....	132
H2510	Exercises.....	133
2511	The comparison test; dominance.....	134
2512	The ratio test.....	135
2513	Exercises.....	138
2514	Limit form for the comparison test.....	139

CONTENTS

ix

2515	Exercises.....	142
2516	Series of arbitrary sign; alternating series.....	143
2517	Absolute and conditional convergence.....	145
2518	Exercises.....	148
H2519	Kummer's, Raabe's, and Gauss's tests.....	149
H2520	Exercises.....	153

26 POWER SERIES AND EXPANSIONS OF FUNCTIONS

2601	Power series.....	155
2602	Interval and radius of convergence.....	156
2603	Exercises.....	159
2604	Taylor series.....	160
2605	Taylor's formula with a remainder.....	163
2606	The integral form of the remainder.....	164
2607	Expansions of functions.....	165
2608	Exercises.....	166
H2609	Continuity within the interval of convergence.....	167
H2610	Exercises.....	168
2611	The Maclaurin series for e^x , $\sin x$, and $\cos x$	169
2612	The Maclaurin series for $\ln(1+x)$; the binomial series.....	171
2613	Elementary operations with power series.....	173
2614	Substitution of power series.....	176
2615	Exercises.....	178
2616	Differentiation and integration of series.....	179
2617	Approximations and computations.....	182
2618	Exercises.....	186
H2619	Proofs of term-by-term operations.....	187
H2620	Exercises.....	189

27 POLAR COORDINATES

2701	Curves in polar coordinates.....	191
2702	Exercises.....	198
2703	Polar equations of conics.....	199
2704	Exercises.....	205
2705	Arc length in polar coordinates.....	206
2706	Exercises.....	207
2707	Area in polar coordinates.....	208
2708	Exercises.....	212
H2709	Area of the set between two graphs.....	213
H2710	Exercises.....	214

28 VECTORS AND MATRICES IN THE PLANE

2801	Vectors in two dimensions	217
2802	Linear combinations of vectors	218
2803	Linear dependence and independence	220
2804	Exercises	225
2805	Direction angles, cosines, and numbers	228
2806	The scalar or inner or dot product	230
2807	Continuity and differentiation of vectors	234
2808	Exercises	237
H2809	Tangential and normal components of acceleration	239
H2810	Exercises	241
2811	Matrices and their linear combinations	241
2812	Linear transformations and products of matrices	244
2813	Algebra of matrices	248
2814	Exercises	250
2815	Determinants	252
2816	Systems of linear equations; Cramer's rule	254
2817	Inverse of a matrix	259
2818	Exercises	261
H2819	Images and preimages	264
H2820	Exercises	266
2821	The general equation of the second degree	266
2822	Translations and rotations	269
2823	The general equation and conic sections	274
2824	Invariants	277
2825	Exercises	283
2826	Simplifications by translations	285
2827	Simplifications by rotations	287
2828	Exercises	292
H2829	Transforming a transformation	294
H2830	Exercises	296

29 VECTORS AND MATRICES IN SPACE

2901	Vectors in space	298
2902	Linear combinations of vectors	300
2903	Exercises	304
2904	Direction cosines; the scalar product	306
2905	Exercises	308
2906	Equations of planes and lines	309
2907	Distance between a plane and a point	314
2908	Exercises	314
H2909	Inner-product spaces	317

CONTENTS

xi

H2910	Exercises.....	321
2911	Vectors orthogonal to two vectors.....	321
2912	Exercises.....	323
2913	Space curves.....	325
2914	Exercises.....	327
2915	Spheres, cylinders, surfaces of revolution.....	328
2916	Exercises.....	332
2917	The standard quadric surfaces.....	333
2918	Exercises.....	338
H2919	Elimination of a variable.....	339
H2920	Exercises.....	341
2921	Translations.....	342
2922	Rotations.....	343
2923	Exercises.....	346
2924	The vector or outer or cross product.....	346
2925	Exercises.....	349
2926	Triple products.....	350
2927	Derivatives of vector functions.....	352
2928	Exercises.....	353
H2929	Curvature and torsion.....	354
H2930	Exercises.....	357

30 FUNCTIONS OF SEVERAL VARIABLES

3001	Point sets in the plane.....	359
3002	Point sets in higher dimensions.....	363
3003	Exercises.....	363
3004	Functions, continuity, and limits.....	364
3005	Continuity and limit theorems.....	368
3006	Exercises.....	370
3007	Iterated limits.....	371
3008	Exercises.....	373
H3009	More about open, closed, and compact sets.....	373
H3010	Exercises.....	377

31 PARTIAL DIFFERENTIATION

3101	Partial derivatives.....	380
3102	Geometrical interpretation.....	381
3103	Partial derivatives of higher order.....	382
3104	Exercises.....	383
3105	Fundamental increment formula.....	384

3106	Differentials.....	386
3107	Change of variables; the chain rule.....	387
3108	Exercises.....	390
H3109	Equality of mixed partial derivatives.....	392
H3110	Exercises.....	393
3111	Implicit functions.....	394
3112	Notational ambiguities.....	395
3113	Exercises.....	397
3114	Directional derivatives and gradients in the plane.....	398
3115	Exercises.....	402
3116	Directional derivatives and gradients in space.....	403
3117	Normal directions to a surface.....	404
3118	Exercises.....	405
H3119	Differentiation under an integral sign.....	406
H3120	Exercises.....	407
3121	The law of the mean.....	408
3122	The extended law of the mean.....	410
3123	Approximations by differentials.....	412
3124	Exercises.....	414
3125	Cylindrical and spherical coordinates.....	415
3126	Exercises.....	418
3127	Several functions defined implicitly.....	418
3128	Exercises.....	422
H3129	A special case of the implicit theorem.....	423
H3130	Exercises.....	426

32 EXTREMA FOR FUNCTIONS OF SEVERAL VARIABLES

3201	Local extrema for functions of two variables.....	428
3202	Sufficient conditions.....	431
3203	Exercises.....	434
3204	Extrema with one constraint, two variables.....	434
3205	Sufficient conditions.....	438
3206	Exercises.....	441
3207	Extrema with one constraint, n variables.....	442
3208	Exercises.....	444
H3209	Extrema with more than one constraint.....	445
H3210	Exercises.....	447

33 MULTIPLE INTEGRALS

3301	Double integrals.....	448
3302	A definition of area.....	451

CONTENTS

xiii

3303	Volumes of certain sets	457
3304	Iterated integrals	459
3305	Exercises	467
3306	The double integral as the limit of a sum	468
3307	Double integrals in polar coordinates	470
3308	Exercises	476
H3309	Further discussion of area	477
H3310	Exercises	480
3311	Mass and density in the plane	481
3312	First moments and centers of mass	482
3313	Exercises	489
3314	Moments of inertia in the plane	490
3315	Exercises	492
3316	Triple integrals in rectangular coordinates; volume	493
3317	Iterated integrals	494
3318	Exercises	496
H3319	Line integrals	497
H3320	Exercises	500
3321	The triple integral as the limit of a sum; cylindrical coordinates	501
3322	Exercises	503
3323	Spherical coordinates	504
3324	Exercises	506
3325	Moments and centers of mass	506
3326	Exercises	509
3327	Surface area	510
3328	Exercises	513
H3329	Surface integrals	514
H3330	Exercises	515

34 DIFFERENTIAL EQUATIONS

3401	The general problem	516
3402	Homogeneous equations	517
3403	Exercises	520
3404	Exact equations; integrating factors	520
3405	Exercises	524
3406	Linear equations of the first order	524
3407	General solution	526
3408	Exercises	527
H3409	Series solutions	528
H3410	Exercises	529
3411	Complex numbers	530
3412	Geometrical representation	532
3413	Polar form; conjugates	533
3414	Exercises	537

3415	Limits and continuity.....	538
3416	Exercises.....	540
3417	Complex-valued functions of a real variable.....	540
3418	Exercises.....	543
H3419	Representation of complex numbers by matrices.....	543
H3420	Exercises.....	544
3421	Linear differential equations of the second order.....	545
3422	Homogeneous equations, real characteristic roots.....	547
3423	Homogeneous equations, imaginary roots.....	550
3424	Exercises.....	552
3425	Nonhomogeneous equations.....	554
3426	Exercises.....	556
3427	Applications.....	557
3428	Exercises.....	562
H3429	Linear dependence and independence.....	563
H3430	Exercises.....	565

APPENDIX

Table I	Squares, cubes, roots, reciprocals.....	568
Table II	Four-place common logarithms.....	570
Table III	Natural logarithms.....	572
Table IV	Numerical constants.....	574
Table V	Exponential function.....	575
Table VI	Natural trigonometric functions, degree measure.....	579
Table VII	Natural trigonometric functions, radian measure.....	580
Table VIII	Degrees, minutes, and seconds to radians.....	581
Table IX	Radians to degrees, minutes, and seconds.....	581
Table X	Formulas from geometry.....	582
Table XI	Greek alphabet.....	582
Table XII	Trigonometric identities.....	583
Table XIII	Derivatives.....	584
Table XIV	Indefinite integrals.....	585

ANSWERS AND HINTS.....	594
-------------------------------	------------

SPECIAL SYMBOLS.....	640
-----------------------------	------------

INDEX.....	645
-------------------	------------

22

Techniques of Integration

2201 INTEGRATION OF THE TRIGONOMETRIC FUNCTIONS

As we have already observed (§1611), the formulas for differentiation of the six standard trigonometric functions can be immediately reformulated as formulas of integration:

$$(1) \quad \int \sin x \, dx = -\cos x + C.$$

$$(2) \quad \int \cos x \, dx = \sin x + C.$$

$$(3) \quad \int \sec^2 x \, dx = \tan x + C.$$

$$(4) \quad \int \csc^2 x \, dx = -\cot x + C.$$

$$(5) \quad \int \sec x \tan x \, dx = \sec x + C.$$

$$(6) \quad \int \csc x \cot x \, dx = -\csc x + C.$$

Of these, (1) and (2) are formulas for integrating two of the six trigonometric functions. In this section we shall derive the following formulas for integrating the remaining four functions:

$$\text{I. } \int \tan x \, dx = \ln |\sec x| + C = -\ln |\cos x| + C.$$

$$\text{II. } \int \cot x \, dx = \ln |\sin x| + C = -\ln |\csc x| + C.$$

$$\text{III. } \int \sec x \, dx = \ln |\sec x + \tan x| + C = -\ln |\sec x - \tan x| + C.$$

$$\text{IV. } \int \csc x \, dx = \ln |\csc x - \cot x| + C = -\ln |\csc x + \cot x| + C.$$

These formulas are all valid for any interval of continuity of the integrand, and by the principle of integration by substitution (§1613) they are also valid if the variable x is replaced by a differentiable function u .

In each of the formulas I–IV, the two expressions on the right are equal since the quantities within absolute values are reciprocals (for example, the product of $\sec x + \tan x$ and $\sec x - \tan x$ is $\sec^2 x - \tan^2 x = 1$), and the values of $\ln$ at two positive numbers that are reciprocals are negatives of each other. In the following derivations we therefore restrict our attention to the *first* of the two given primitives in each case. The simplest way to prove I–IV is undoubtedly to differentiate the quantities on the right and verify that the resulting derivatives are the appropriate integrands on the left. Although this method provides perfectly valid proofs (show the details in Ex. 47, §2203), it is psychologically unsatisfying in that it merely confirms a result already achieved. We shall now show how we can start with the integrals on the left and *derive* the formulas on the right without knowing them in advance.

Derivation of I. Write the integrand in terms of $\sin x$ and $\cos x$, and use the substitution $u = \cos x$, $du = -\sin x \, dx$:

$$(7) \quad \int \tan x \, dx = \int \frac{\sin x \, dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x} = -\int \frac{du}{u}.$$

By the formula $\int \frac{du}{u} = \ln |u| + C$ (I, §1717), with $u = \cos x$, this integral is equal to

$$-\ln |u| + C = -\ln |\cos x| + C = \ln |\sec x| + C.$$

Derivation of II. This is similar to I and is left to Exercise 48, §2203.

Derivation of III. This derivation is more subtle. It turns out that sines and cosines are no help. The two reasonably logical substitutions $u = \sec x$ and $u = \tan x$ also lead to frustration. For example, with $u = \sec x$, $du = \sec x \tan x \, dx$,

$$(8) \quad \int \sec x \, dx = \int \frac{\sec x \tan x \, dx}{\tan x} = \int \frac{du}{\tan x} = \int \frac{du}{\pm \sqrt{u^2 - 1}},$$

and we are led to an integral for which we have, as yet, no formula. It is not unreasonable to try next a linear combination of $\sec x$ and $\tan x$, and either $\sec x + \tan x$ or $\sec x - \tan x$ gives a manageable result. For example, with

$u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)dx = (\sec x)(\sec x + \tan x)dx = (\sec x)u dx$, so that

$$(9) \quad \int \sec x dx = \int \frac{(\sec x)u dx}{u} = \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C.$$

Derivation of IV. This is similar to III, and is left to Exercise 49, §2203.

Example 1. $\int \cot 5x dx = \frac{1}{5} \int \cot 5x d(5x) = \frac{1}{5} \ln |\sin 5x| + C,$

$$\int x \sec x^2 dx = \frac{1}{2} \int \sec x^2 d(x^2) = \frac{1}{2} \ln |\sec x^2 + \tan x^2| + C,$$

$$\int_0^\pi \tan \frac{x}{4} dx = 4 \int_0^\pi \tan \frac{x}{4} d\left(\frac{x}{4}\right) = 4 \ln \left| \sec \frac{x}{4} \right| \Big|_0^\pi = 4(\ln \sqrt{2} - \ln 1) = \ln \sqrt{2^4} = \ln 4,$$

$$\begin{aligned} \int_0^\pi \csc \frac{x+\pi}{3} dx &= 3 \int_0^\pi \csc \frac{x+\pi}{3} d\left(\frac{x+\pi}{3}\right) = 3 \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \csc u du = 3 \ln |\csc u - \cot u| \Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \\ &= 3 \left(\ln \left| \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right| - \ln \left| \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \right| \right) = 3 \ln \left(\frac{3}{\sqrt{3}} / \frac{1}{\sqrt{3}} \right) = 3 \ln 3 = \ln 27. \end{aligned}$$

The trigonometric product formulas ((13)–(16), §717, reproduced in 6 of Table XII, page 583) provide a means of integrating certain functions formed as products of sines, as products of cosines, or as products of sines and cosines, as illustrated below. For convenience we adapt these formulas to a form more suitable for present purposes, by means of a shift in notation:

$$(10) \quad \sin ax \cos bx = \frac{1}{2} \sin(a+b)x + \frac{1}{2} \sin(a-b)x,$$

$$(11) \quad \cos ax \sin bx = \frac{1}{2} \sin(a+b)x - \frac{1}{2} \sin(a-b)x,$$

$$(12) \quad \cos ax \cos bx = \frac{1}{2} \cos(a+b)x + \frac{1}{2} \cos(a-b)x,$$

$$(13) \quad \sin ax \sin bx = -\frac{1}{2} \cos(a+b)x + \frac{1}{2} \cos(a-b)x.$$

Example 2. Integrate: $\int \sin 3x \cos 5x dx$.

Solution. We use (11) rather than (10) since it is more convenient if $a-b$ is positive than it is if $a-b$ is negative. Accordingly, with $a=5$ and $b=3$, we have

$$\begin{aligned} \int \cos 5x \sin 3x dx &= \frac{1}{2} \int \sin 8x dx - \frac{1}{2} \int \sin 2x dx \\ &= \frac{1}{16} \int \sin 8x d(8x) - \frac{1}{4} \int \sin 2x d(2x) = \frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + C. \end{aligned}$$

Formula (10) leads to the same result since $\sin(-2x) = -\sin 2x$.

2202 POWERS OF SINES AND COSINES

An integral of the form

$$(1) \quad \int \sin^m x \cos^n x dx, \text{ where at least one of the exponents } m \text{ and } n \text{ is a positive odd integer,}$$