CALCULUS WITH ANALYTIC GEOMETRY

2

JOHN M. H. OLMSTED

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Techniques of Integration

2201 INTEGRATION OF THE TRIGONOMETRIC FUNCTIONS

As we have already observed (§1611), the formulas for differentiation of the six standard trigonometric functions can be immediately reformulated as formulas of integration:

(1) $$\int \sin x \, dx = -\cos x + C.$$

(2) $$\int \cos x \, dx = \sin x + C.$$

(3) $$\int \sec^2 x \, dx = \tan x + C.$$

(4) $$\int \csc^2 x \, dx = -\cot x + C.$$

(5) $$\int \sec x \tan x \, dx = \sec x + C.$$

(6) $$\int \csc x \cot x \, dx = -\csc x + C.$$

Of these, (1) and (2) are formulas for integrating two of the six trigonometric functions. In this section we shall derive the following formulas for integrating the remaining four functions:

$$\textbf{I.}\quad \int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C.$$

$$\textbf{II.}\quad \int \cot x\,dx = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\textbf{III.}\quad \int \sec x\,dx = \ln|\sec x + \tan x| + C = -\ln|\sec x - \tan x| + C.$$

$$\textbf{IV.}\quad \int \csc x\,dx = \ln|\csc x - \cot x| + C = -\ln|\csc x + \cot x| + C.$$

These formulas are all valid for any interval of continuity of the integrand, and by the principle of integration by substitution (§1613) they are also valid if the variable x is replaced by a differentiable function u.

In each of the formulas I–IV, the two expressions on the right are equal since the quantities within absolute values are reciprocals (for example, the product of $\sec x + \tan x$ and $\sec x - \tan x$ is $\sec^2 x - \tan^2 x = 1$), and the values of ln at two positive numbers that are reciprocals are negatives of each other. In the following derivations we therefore restrict our attention to the *first* of the two given primitives in each case. The simplest way to prove I–IV is undoubtedly to differentiate the quantities on the right and verify that the resulting derivatives are the appropriate integrands on the left. Although this method provides perfectly valid proofs (show the details in Ex. 47, §2203), it is psychologically unsatisfying in that it merely confirms a result already achieved. We shall now show how we can start with the integrals on the left and *derive* the formulas on the right without knowing them in advance.

Derivation of I. Write the integrand in terms of $\sin x$ and $\cos x$, and use the substitution $u = \cos x$, $du = -\sin x\,dx$:

$$(7)\qquad \int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x} = -\int \frac{du}{u}.$$

By the formula $\int \frac{du}{u} = \ln|u| + C$ (I, §1717), with $u = \cos x$, this integral is equal to

$$-\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C.$$

Derivation of II. This is similar to I and is left to Exercise 48, §2203.

Derivation of III. This derivation is more subtle. It turns out that sines and cosines are no help. The two reasonably logical substitutions $u = \sec x$ and $u = \tan x$ also lead to frustration. For example, with $u = \sec x$, $du = \sec x \tan x\,dx$,

$$(8)\qquad \int \sec x\,dx = \int \frac{\sec x \tan x\,dx}{\tan x} = \int \frac{du}{\tan x} = \int \frac{du}{\pm\sqrt{u^2 - 1}},$$

and we are led to an integral for which we have, as yet, no formula. It is not unreasonable to try next a linear combination of $\sec x$ and $\tan x$, and either $\sec x + \tan x$ or $\sec x - \tan x$ gives a manageable result. For example, with

$u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)dx = (\sec x)(\sec x + \tan x)dx = (\sec x)u\,dx$, so that

$$(9)\quad \int \sec x\,dx = \int \frac{(\sec x)u\,dx}{u} = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

Derivation of IV. This is similar to III, and is left to Exercise 49, §2203.

Example 1. $\displaystyle\int \cot 5x\,dx = \frac{1}{5}\int \cot 5x\,d(5x) = \frac{1}{5}\ln|\sin 5x| + C,$

$$\int x \sec x^2\,dx = \frac{1}{2}\int \sec x^2\,d(x^2) = \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C,$$

$$\int_0^\pi \tan\frac{x}{4}\,dx = 4\int_0^\pi \tan\frac{x}{4}\,d\left(\frac{x}{4}\right) = 4\ln\left|\sec\frac{x}{4}\right|\Bigg|_0^\pi = 4(\ln\sqrt{2} - \ln 1) = \ln\sqrt{2}^4 = \ln 4,$$

$$\int_0^\pi \csc\frac{x+\pi}{3}\,dx = 3\int_0^\pi \csc\frac{x+\pi}{3}\,d\left(\frac{x+\pi}{3}\right) = 3\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \csc u\,du = 3\ln|\csc u - \cot u|\Bigg|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= 3\left(\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right|\right) = 3\ln\left(\frac{3}{\sqrt{3}}\Big/\frac{1}{\sqrt{3}}\right) = 3\ln 3 = \ln 27.$$

The trigonometric product formulas ((13)–(16), §717, reproduced in 6 of Table XII, page 583) provide a means of integrating certain functions formed as products of sines, as products of cosines, or as products of sines and cosines, as illustrated below. For convenience we adapt these formulas to a form more suitable for present purposes, by means of a shift in notation:

$$(10)\qquad \sin ax \cos bx = \tfrac{1}{2}\sin(a+b)x + \tfrac{1}{2}\sin(a-b)x,$$

$$(11)\qquad \cos ax \sin bx = \tfrac{1}{2}\sin(a+b)x - \tfrac{1}{2}\sin(a-b)x,$$

$$(12)\qquad \cos ax \cos bx = \tfrac{1}{2}\cos(a+b)x + \tfrac{1}{2}\cos(a-b)x,$$

$$(13)\qquad \sin ax \sin bx = -\tfrac{1}{2}\cos(a+b)x + \tfrac{1}{2}\cos(a-b)x.$$

Example 2. Integrate: $\displaystyle\int \sin 3x \cos 5x\,dx.$

Solution. We use (11) rather than (10) since it is more convenient if $a - b$ is positive than it is if $a - b$ is negative. Accordingly, with $a = 5$ and $b = 3$, we have

$$\int \cos 5x \sin 3x\,dx = \frac{1}{2}\int \sin 8x\,dx - \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{16}\int \sin 8x\,d(8x) - \frac{1}{4}\int \sin 2x\,d(2x) = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C.$$

Formula (10) leads to the same result since $\sin(-2x) = -\sin 2x$.

2202 POWERS OF SINES AND COSINES

An integral of the form

(1) $\displaystyle\int \sin^m x \cos^n x\,dx$, *where at least one of the exponents m and n is a positive odd integer,*