THE CALCULUS
WITH ANALYTIC GEOMETRY
third edition

Part I Functions of one variable, plane analytic geometry, and infinite series

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To Gordon Marc

Preface

This third edition of THE CALCULUS WITH ANALYTIC GEOMETRY, like the other two, is designed for prospective mathematics majors as well as for students whose primary interest is in engineering, the physical sciences, or nontechnical fields. A knowledge of high-school algebra and geometry is assumed.

The text is available either in one volume or in two parts: Part I consists of the first sixteen chapters, and Part II comprises Chapters 16 through 21 (Chapter 16 on Infinite Series is included in both parts to make the use of the two-volume set more flexible). The material in Part I consists of the differential and integral calculus of functions of a single variable and plane analytic geometry, and it may be covered in a one-year course of nine or ten semester hours or twelve quarter hours. The second part is suitable for a course consisting of five or six semester hours or eight quarter hours. It includes the calculus of several variables and a treatment of vectors in the plane, as well as in three dimensions, with a vector approach to solid analytic geometry.

The objectives of the previous editions have been maintained. I have endeavored to achieve a healthy balance between the presentation of elementary calculus from a rigorous approach and that from the older, intuitive, and computational point of view. Bearing in mind that a textbook should be written for the student, I have attempted to keep the presentation geared to a beginner's experience and maturity and to leave no step unexplained or omitted. I desire that the reader be aware that proofs of theorems are necessary and that these proofs be well motivated and carefully explained so that they are understandable to the student who has achieved an average mastery of the preceding sections of the book. If a theorem is stated without proof, I have generally augmented the discussion by both figures and examples, and in such cases I have always stressed that what is presented is an illustration of the content of the theorem and is not a proof.

Changes in the third edition occur in the first five chapters. The first section of Chapter 1 has been rewritten to give a more detailed exposition of the real-number system. The introduction to analytic geometry in this chapter includes the traditional material on straight lines as well as that of the circle, but a discussion of the parabola is postponed to Chapter 14, The Conic Sections. Functions are now introduced in Chapter 1. I have defined a function as a set of ordered pairs and have used this idea to point up the concept of a function as a correspondence between sets of real numbers.

The treatment of limits and continuity which formerly consisted of ten sections in Chapter 2 is now in two chapters (2 and 4), with the chapter on the derivative placed between them. The concepts of limit and continuity are at the heart of any first course in the calculus. The notion of a limit of a function is first given a step-by-step motivation, which brings the discussion from computing the value of a function near a number, through an intuitive treatment of the limiting process, up to a rigorous epsilon-delta definition. A sequence of examples progressively graded in difficulty is included. All the limit theorems are stated, and some proofs are presented in the text, while other proofs have been outlined in the exercises. In the discussion of continuity, I have used as examples and counterexamples "common, everyday" functions and have avoided those that would have little intuitive meaning.

In Chapter 3, before giving the formal definition of a derivative, I have defined the tangent line to a curve and instantaneous velocity in rectilinear motion in order to demonstrate in advance that the concept of a derivative is of wide application, both geometrical and physical. Theorems on differentiation are proved and illustrated by examples. Application of the derivative to related rates is included.

Additional topics on limits and continuity are given in Chapter 4. Continuity on a closed interval is defined and discussed, followed by the introduction of the Extreme-Value Theorem, which involves such functions. Then the Extreme-Value Theorem is used to find the absolute extrema of functions continuous on a closed interval. Chapter 4 concludes with Rolle's Theorem and the Mean-Value Theorem. Chapter 5 gives additional applications of the derivative, including problems on curve sketching as well as some related to business and economics.

The antiderivative is treated in Chapter 6. I use the term "antidifferentiation" instead of indefinite integration, but the standard notation $\int f(x)\, dx$ is retained so that you are not given a bizarre new notation that would make the reading of standard references difficult. This notation will suggest that some relation must exist between definite integrals, introduced in Chapter 7, and antiderivatives, but I see no harm in this as long as the presentation gives the theoretically proper view of the definite integral as the limit of sums. Exercises involving the evaluation of definite integrals by finding limits of sums are given in Chapter 7 to stress that this is how they are calculated. The introduction of the definite integral follows the definition of the measure of the area under a curve as a limit of sums. Elementary properties of the definite integral are derived and the fundamental theorem of the calculus is proved. It is emphasized that this is a theorem, and an important one, because it provides us with an alternative to computing limits of sums. It is also emphasized that the definite integral is in no sense some special type of antiderivative. In Chapter 8 I have given numerous applications of definite integrals. The presentation highlights not only the manipulative techniques but also the fundamental principles involved. In each application, the definitions of the new terms are intuitively motivated and explained.

The treatment of logarithmic and exponential functions in Chapter 9 is the modern approach. The natural logarithm is defined as an integral, and after the discussion of the inverse of a function, the exponential function is defined as the inverse of the natural logarithmic function. An irrational power of a real number is then defined. The trigonometric functions are defined in Chapter 10 as functions assigning numbers to numbers. The important trigonometric identities are derived and used to obtain the formulas for the derivatives and integrals of these functions. Following are sections on the differentiation and integration of the trigonometric functions as well as of the inverse trigonometric functions.

Chapter 11, on techniques of integration, involves one of the most important computational aspects of the calculus. I have explained the theoretical backgrounds of each different method after an introductory motivation. The mastery of integration techniques depends upon the examples, and I have used as illustrations problems that the student will certainly meet in practice, those which require patience and persistence to solve. The material on the approximation of definite integrals includes the statement of theorems for computing the bounds of the error involved in these approximations. The theorems and the problems that go with them, being self-contained, can be omitted from a course if the instructor so wishes.

A self-contained treatment of hyperbolic functions is in Chapter 12. This chapter may be studied immediately following the discussion of the circular trigonometric functions in Chapter 10, if so desired. The geometric interpretation of the hyperbolic functions is postponed until Chapter 17 because it involves the use of parametric equations.

Polar coordinates and some of their applications are given in Chapter 13. In Chapter 14, conics are treated as a unified subject to stress their natural and close relationship to each other. The parabola is discussed in the first two sections. Then equations of the conics in polar coordinates are treated, and the cartesian equations of the ellipse and the hyperbola are derived from the polar equations. The topics of indeterminate forms, improper integrals, and Taylor's formula, and the computational techniques involved, are presented in Chapter 15.

I have attempted in Chapter 16 to give as complete a treatment of infinite series as is feasible in an elementary calculus text. In addition to the customary computational material, I have included the proof of the equivalence of convergence and boundedness of monotonic sequences based on the completeness property of the real numbers and the proofs of the computational processes involving differentiation and integration of power series.

The first five sections of Chapter 17 on vectors in the plane can be taken up after Chapter 5 if it is desired to introduce vectors earlier in the course. The approach to vectors is modern, and it serves both as an introduction to the viewpoint of linear algebra and to that of classical vector analysis. The applications are to physics and geometry. Chapter 18 treats vectors in three-dimensional space, and, if desired, the topics in the first three sections of this chapter may be studied concurrently with the corresponding topics in Chapter 17.

Limits, continuity, and differentiation of functions of several variables are considered in Chapter 19. The discussion and examples are applied mainly to functions of two and three variables; however, statements of most of the definitions and theorems are extended to functions of n variables.

In Chapter 20, a section on directional derivatives and gradients is followed by a section that shows the application of the gradient to finding an equation of the tangent plane to a surface. Applications of partial derivatives to the solution of extrema problems and an introduction to Lagrange multipliers are presented, as well as a section on applications of partial derivatives in economics. Three sections, new in the third edition, are devoted to line integrals and related topics. The double integral of a function of two variables and the triple integral of a function of three variables, along with some applications to physics, engineering, and geometry, are given in Chapter 21.

New to this edition is a short table of integrals appearing on the front and back endpapers. However, as stated in Chapter 11, you are advised to use a table of integrals only after you have mastered integration.

Louis Leithold

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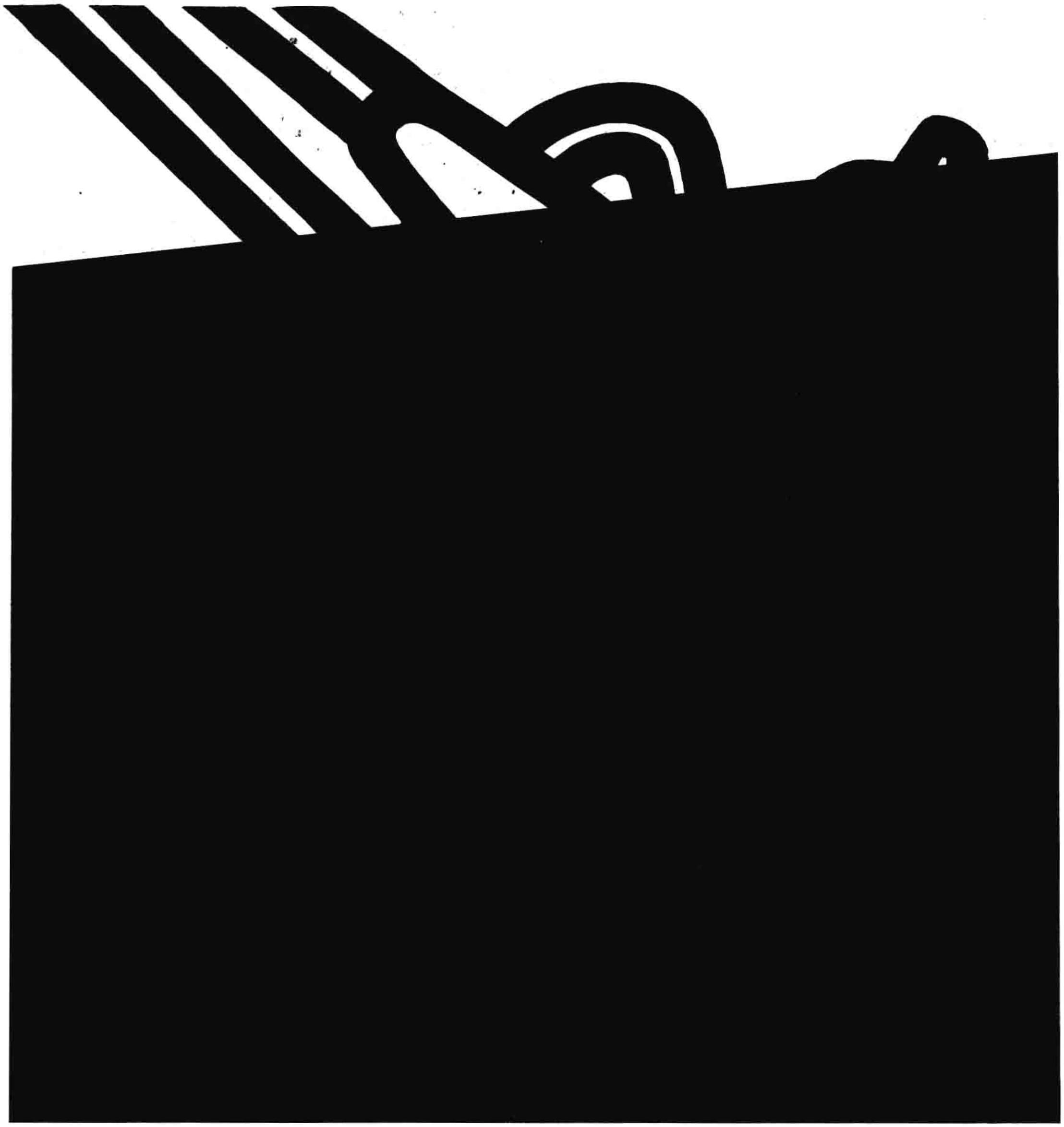

Real numbers, introduction to analytic geometry, and functions

1.1 SETS, REAL NUMBERS, AND INEQUALITIES

The idea of "set" is used extensively in mathematics and is such a basic concept that it is not given a formal definition. We can say that a *set* is a collection of objects, and the objects in a set are called the *elements* of a set. We may speak of the set of books in the New York Public Library, the set of citizens of the United States, the set of trees in Golden Gate Park, and so on. In calculus, we are concerned with the set of real numbers. Before discussing this set, we introduce some notation and definitions.

We want every set to be *well defined;* that is, there should be some rule or property that enables one to decide whether a given object is or is not an element of a specific set. A pair of braces { } used with words or symbols can describe a set.

If S is the set of natural numbers less than 6, we can write the set S as

$$\{1, 2, 3, 4, 5\}$$

We can also write the set S as

$$\{x, \text{ such that } x \text{ is a natural number less than } 6\}$$

where the symbol "x" is called a "variable." A *variable* is a symbol used to represent any element of a given set. Another way of writing the above set S is to use what is called *set-builder notation*, where a vertical bar is used in place of the words "such that." Using set-builder notation to describe the set S, we have

$$\{x | x \text{ is a natural number less than } 6\}$$

which is read "the set of all x such that x is a natural number less than 6."

The set of natural numbers will be denoted by N. Therefore, we may write the set N as

$$\{1, 2, 3, \ldots\}$$

where the three dots are used to indicate that the list goes on and on with no last number. With set-builder notation the set N may be written as $\{x | x \text{ is a natural number}\}$.

The symbol "$\in$" is used to indicate that a specific element belongs to a set. Hence, we may write $8 \in N$, which is read "8 is an element of N." The notation $a, b \in S$ indicates that both a and b are elements of S. The symbol $\notin$ is read "is not an element of." Thus, we read $\frac{1}{2} \notin N$ as "$\frac{1}{2}$ is not an element of N."

We denote the set of all integers by J. Because every element of N is also an element of J (that is, every natural number is an integer), we say that N is a "subset" of J, written $N \subseteq J$.

1.1.1 Definition The set S is a *subset* of the set T, written $S \subseteq T$, if and only if every element of S is also an element of T. If, in addition, there is at least one element of T

which is not an element of S, then S is a *proper subset* of T, and it is written $S \subset T$.

Observe from the definition that every set is a subset of itself, but a set is *not* a proper subset of itself.

In Definition 1.1.1, the "if and only if" qualification is used to combine two statements: (i) "the set S is a subset of the set T *if* every element of S is also an element of T"; and (ii) "the set S is a subset of set T *only if* every element of S is also an element of T," which is logically equivalent to the statement "if S is a subset of T, then every element of S is also an element of T."

• ILLUSTRATION 1: Let N be the set of natural numbers and let M be the set denoted by $\{x | x$ is a natural number less than $10\}$. Because every element of M is also an element of N, M is a subset of N and we write $M \subseteq N$. Also, there is at least one element of N which is not an element of M, and so M is a proper subset of N and we may write $M \subset N$. Furthermore, because $\{6\}$ is the set consisting of the number 6, $\{6\} \subset M$, which states that the set consisting of the single element 6 is a proper subset of the set M. We may also write $6 \in M$, which states that the number 6 is an element of the set M. •

Consider the set $\{x | 2x + 1 = 0$, and $x \in J\}$. This set contains no elements because there is no integer solution of the equation $2x + 1 = 0$. Such a set is called the "empty set" or the "null set."

1.1.2 Definition The *empty set* (or *null set*) is the set that contains no elements. The empty set is denoted by the symbol $\varnothing$.

The concept of "subset" may be used to define what is meant by two sets being "equal."

1.1.3 Definition Two sets A and B are said to be *equal*, written $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Essentially, this definition states that the two sets A and B are equal if and only if every element of A is an element of B and every element of B is an element of A, that is, if the sets A and B have identical elements.

There are two operations on sets that we shall find useful as we proceed. These operations are given in Definitions 1.1.4 and 1.1.5.

1.1.4 Definition Let A and B be two sets. The *union* of A and B, denoted by $A \cup B$ and read "A union B," is the set of all elements that are in A or in B or in both A and B.

EXAMPLE 1: Let $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{1, 4, 9, 16\}$, and $C = \{2, 10\}$. Find

(a) $A \cup B$ (b) $A \cup C$
(c) $B \cup C$ (d) $A \cup A$

SOLUTION:

(a) $A \cup B = \{1, 2, 4, 6, 8, 9, 10, 12, 16\}$
(b) $A \cup C = \{2, 4, 6, 8, 10, 12\}$
(c) $B \cup C = \{1, 2, 4, 9, 10, 16\}$
(d) $A \cup A = \{2, 4, 6, 8, 10, 12\} = A$

1.1.5 Definition Let A and B be two sets. The *intersection* of A and B, denoted by $A \cap B$ and read "A intersection B," is the set of all elements that are in both A and B.

EXAMPLE 2: If A, B, and C are the sets defined in Example 1, find

(a) $A \cap B$ (b) $A \cap C$
(c) $B \cap C$ (d) $A \cap A$

SOLUTION:

(a) $A \cap B = \{4\}$ (b) $A \cap C = \{2, 10\}$
(c) $B \cap C = \varnothing$ (d) $A \cap A = \{2, 4, 6, 8, 10, 12\} = A$

The *real number system* consists of a set of elements called *real numbers* and two operations called *addition* and *multiplication*. The set of real numbers is denoted by R^1. The operation of addition is denoted by the symbol "+", and the operation of multiplication is denoted by the symbol "·". If $a, b \in R^1$, $a + b$ denotes the *sum* of a and b, and $a \cdot b$ (or ab) denotes their *product*.

We now present seven axioms that give laws governing the operations of addition and multiplication on the set R^1. The word *axiom* is used to indicate a formal statement that is assumed to be true without proof.

1.1.6 Axiom (*Closure and Uniqueness Laws*) If $a, b \in R^1$, then $a + b$ is a unique real number, and ab is a unique real number.

1.1.7 Axiom (*Commutative Laws*) If $a, b \in R^1$, then

$$a + b = b + a \quad \text{and} \quad ab = ba$$

1.1.8 Axiom (*Associative Laws*) If $a, b, c \in R^1$, then

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a(bc) = (ab)c$$

1.1.9 Axiom (*Distributive Law*) If $a, b, c \in R^1$, then

$$a(b + c) = ab + ac$$

1.1.10 Axiom (*Existence of Negative Elements*) There exist two distinct real numbers 0 and 1 such that for any real number a,

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$