problems in quantum mechanics

editor D ter Haar

third edition

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Preface to second edition

This is essentially an enlarged and revised second edition of a collection of problems which consisted of a text by Gol'dman and Krivchenkov augmented by a selection from a similar text by Kogan and Galitskii.

In preparing the present edition I have used the opportunity to revise some of the problems in the first edition, to change a few of the solutions, and to make the notation both uniform and conforming to English usage. Also, I have added a few problems from a collection by Irodov on atomic physics and a number of new problems which were mainly taken from Oxford University Examination papers. I should like to express my thanks to the Oxford University Press for permission to include these problems.

These problems can be used either in conjunction with any modern textbook, such as those by Schiff, Kramers, Landau and Lifshitz, Messiah, or Davydov, or as advanced reading for anybody who is familiar with the basic ideas of quantum mechanics from a more elementary textbook.

Oxford,
September 1963

D. ter Haar

Preface to third edition

In preparing the third edition I have dropped some of the problems, slightly rearranged the order of the problems, and added new problems to the old chapters, as well as added new sections on the density matrix and annihilation and creation operator problems and on relativistic wave equations. Otherwise the aim and scope of the book remain much as they were before, but to help readers I have added stars to more complicated problems.

Oxford,
September 1974

D. ter Haar

1 problems

One-dimensional motion

1. Determine the energy levels and the normalised wave functions of a particle in a "potential well". The potential energy V of the particle is:

$$V = \infty \quad \text{for} \quad x < 0 \text{ and for } x > a;$$

$$V = 0 \quad \text{for} \quad 0 < x < a.$$

2. Show that for particles in a "potential well" (see preceding problem) the following relations hold:

$$\bar{x} = \tfrac{1}{2}a,$$

$$\overline{(x-\bar{x})^2} = \frac{a^2}{12}\left(1 - \frac{6}{n^2\pi^2}\right).$$

Show also that for large values of n the above result agrees with the corresponding classical result.

3. Determine the momentum probability distribution function for particles in the nth energy state in a "potential well".

4. A particle in an infinitely deep rectangular potential well is in a state described by the wave function

$$\psi(x) = Ax(a-x),$$

where a is the well width and A a constant.

Find the probability distribution for the different energies of the particle and also the average value and the dispersion of the energy.

5. A particle is in the ground state in a potential well of length a. At time $t = 0$ the wall at $x = a$ is suddenly moved to $x = 2a$. Calculate the probability that, at time $t > 0$,

(a) the energy of the particle is the same as before $t = 0$; and

(b) the energy of the particle is less than before $t = 0$.

6*. A particle is enclosed in a one-dimensional rectangular potential well with infinitely high walls. Evaluate the average force exerted by the particle on the wall of the well.

7. Determine the energy levels and wave functions of a particle in an asymmetrical potential well (see fig. 1). Consider the case where $V_1 = V_2$.

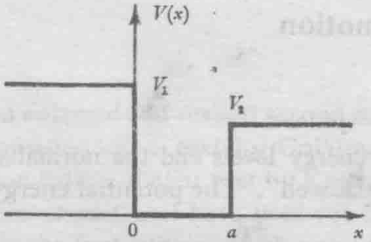

Fig. 1.

8. The Hamiltonian of an oscillator is equal to $\hat{H} = \hat{p}^2/2\mu + \mu\omega^2\hat{x}^2/2$, where $\hat{p}$ and $\hat{x}$ satisfy the commutation relationships $\hat{p}\hat{x} - \hat{x}\hat{p} = -i\hbar$. In order to eliminate $\hbar$, μ, and ω from the calculations, we introduce new variables $\hat{P}$ and $\hat{Q}$,†

$$\hat{P} = \frac{1}{\sqrt{(\mu\hbar\omega)}}\,\hat{p}, \quad \hat{Q} = \sqrt{\left(\frac{\mu\omega}{\hbar}\right)}\,\hat{x} \quad (P\hat{Q} - \hat{Q}P = -i),$$

and the energy E will be expressed in units $\hbar\omega$ $(E = \epsilon\hbar\omega)$. The Schrödinger equation for the oscillator in the new variables will be of the form

$$\hat{H}'\psi = \tfrac{1}{2}(\hat{P}^2 + \hat{Q}^2)\psi = \epsilon\psi.$$

(a) Use the commutation relation $\hat{P}\hat{Q} - \hat{Q}\hat{P} = -i$, to show that

$$\tfrac{1}{2}(\hat{P}^2 + \hat{Q}^2)(\hat{Q} \pm i\hat{P})^n\psi = (\epsilon \mp n)(\hat{Q} \pm i\hat{P})^n\psi.$$

(b) Determine the normalised wave functions and the energy levels of the oscillator.

(c) Determine the commutator of the operator $\hat{a} = (1/\sqrt{2})(\hat{Q} + i\hat{P})$ and its Hermitean conjugate operator $\hat{a}^+ = (1/\sqrt{2})(\hat{Q} - i\hat{P})$. Express the wave function of the nth excited state in terms of the wave function of the ground state using the operator $\hat{a}$.

(d) Determine the matrix elements of the operators $\hat{P}$ and $\hat{Q}$ in the energy representation.

Hint. $\hat{P}^2 + \hat{Q}^2 - 1 = (\hat{P} + i\hat{Q})(\hat{P} - i\hat{Q})$.

9. Using the results of the preceding problem, show by direct multiplication of matrices that for an oscillator in the nth stationary state we have

$$\overline{(\Delta x)^2} = \overline{x^2} = \frac{\hbar}{\mu\omega}(n + \tfrac{1}{2}); \quad \overline{(\Delta p)^2} = \overline{p^2} = \mu\hbar\omega(n + \tfrac{1}{2}).$$

† Operators are indicated by a caret ^.

Show that, for any stationary state, the root-mean-square value of x is the same as it would have been for a classical oscillator with the same energy.

10. A particle moves in a potential $V(x) = \frac{1}{2}\mu\omega^2 x^2$. Determine the probability w to find the particle outside the classical limits, when it is in its ground state.

11. Find the energy levels of a particle moving in a potential of the following form:

$$V(x) = \infty \quad (x < 0); \quad V(x) = \frac{\mu\omega^2 x^2}{2} \quad (x > 0).$$

12. Write down the Schrödinger equation for an oscillator in the "p-representation" and determine the momentum probability distribution function.

13. Find the wave functions and energy levels of a particle in a potential $V(x) = V_0(a/x - x/a)^2$ $(x > 0)$ (see fig. 2) and show that the energy spectrum is the same as the oscillator spectrum.

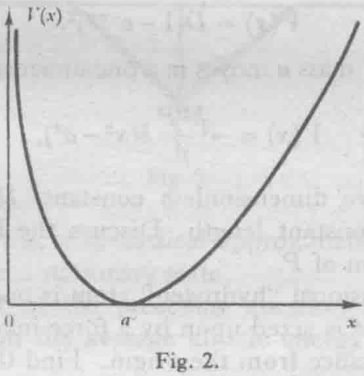

Fig. 2.

14*. Determine the energy levels for a particle in a potential $V = -V_0/\cosh^2(x/a)$ (see fig. 3).

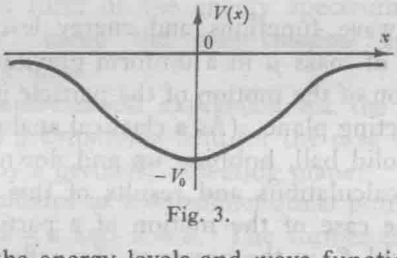

Fig. 3.

15*. Determine the energy levels and wave functions for a particle in the potential $V = V_0 \cot^2(\pi x/a)$ $(0 < x < a)$ (see fig. 4), and derive the normalisation constant of the ground state wave function.

Consider the limiting cases of small and large values of V_0.

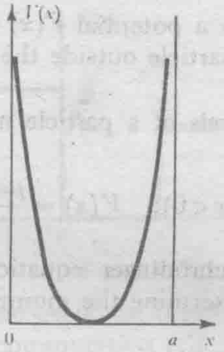

Fig. 4.

16. Determine the energy levels for a particle in a potential

$$V(x) = D(1-e^{-ax})^2.$$

17. An electron of mass μ moves in a one-dimensional potential

$$V(x) = -\frac{\hbar^2 P}{\mu}\,\delta(x^2-a^2),$$

where P is a positive dimensionless constant, $\delta(x)$ the Dirac delta-function, and a a constant length. Discuss the bound states for this potential as a function of P.

18*. A one-dimensional "hydrogen" atom is one in which an electron confined to the x-axis is acted upon by a force inversely proportional to the square of its distance from the origin. Find the energy eigenvalues and the eigenfunctions of this system.

19. Determine the wave functions of a charged particle in a uniform field $V(x) = -Fx$.

20*. Find the wave functions and energy levels of the stationary states of a particle of mass μ in a uniform gravitational field g for the case when the region of the motion of the particle is limited from below by a perfectly reflecting plane. (As a classical analogy of this system we can take a heavy solid ball, bobbing up and down on a metallic plate. We note that all calculations and results of this problem are clearly correct also for the case of the motion of a particle of charge e in a uniform electric field $\mathscr{E}$, in the presence of a reflecting plane, provided we replace in all equations g by $(e/m)\mathscr{E}$.) Take the limit to classical mechanics.

21*. Derive expressions for the wavefunction of a particle moving in a potential $V(x)$ using the semi-classical approximation. Give conditions for the applicability of the approximation and determine the quantization condition.

22. Use the semi-classical approximation to derive an expression for the number of discrete levels of a particle moving in a given potential.

23. Determine in the semi-classical approximation the energy spectrum of a particle in the following potentials:

(a) $V = \frac{1}{2}\mu\omega^2 x^2$ (oscillator);

(b) $V = V_0 \cot^2(\pi x/a)$ $(0 < x < a)$.

24. Use the semi-classical approximation to determine the bound energy levels for a particle of mass μ moving in a potential which equals $-V_0$ for $x = 0$, changes linearly with x until it vanishes at $x = \pm a$, and is zero for $|x| > a$ (see fig. 5). Determine also the total number of discrete energy levels.

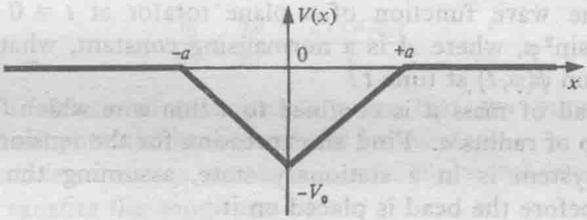

Fig. 5.

25. Determine in the semi-classical approximation the average value of the kinetic energy in a stationary state.

26. Use the result of the preceding question to find in the semi-classical approximation the average kinetic energy of a particle in the following potentials:

(a) $V = \frac{1}{2}\mu\omega^2 x^2$;

(b) $V = V_0 \cot^2(\pi x/a)$ $(0 < x < a)$ (see problem 23).

27. Determine the form of the energy spectrum of a particle in a potential $V(x) = ax^{\nu}$, using the semi-classical approximation and applying the virial theorem.

28. Obtain the semi-classical expression for the energy levels of a particle in a uniform gravitational field for the case where its motion is limited from below by a perfectly reflecting plane.

29. A particle oscillates in a one-dimensional potential field between two turning points $x = a$ and $x = b$. The former is due to a vertical potential wall, while the latter is of the more usual type with dV/dx finite. Apply the WKB method to find the quantisation condition for a stationary state in such a potential.

30*. Determine in the semi-classical approximation the form of the potential energy $V(x)$ for a given energy spectrum E_n. $V(x)$ may be assumed to be an even function $V(x) = V(-x)$, which increases monotonically for $x > 0$.

31*. Find the semi-classical solution of the Schrödinger equation in the momentum-representation.

Show that the same semi-classical function is obtained by going over from the "x-representation" to the "p-representation" starting from the usual semi-classical coordinate wave function.

32. Find the wave functions and energy levels of the stationary states of a plane rotator with moment of inertia I.

A rotator is a system of two rigidly connected particles rotating in a plane (or in space). The moment of inertia of a rotator is equal to $I = \mu a^2$, where μ is the reduced mass of the particles and a their distance apart.

33. If the wave function of a plane rotator at $t = 0$ is given by $\psi(\varphi, 0) = A \sin^2 \varphi$, where A is a normalising constant, what will be the wave function $\psi(\varphi, t)$ at time t?

34. A bead of mass μ is confined to a thin wire which forms a rigid circular loop of radius a. Find an expression for the tension in the wire when the system is in a stationary state, assuming the wire to be unstressed before the bead is placed on it.

35. Write down the Schrödinger equation in the "p-representation" for a particle moving in a periodic potential $V(x) = V_0 \cos bx$.

36. Write down the Schrödinger equation in the "p-representation" for a particle moving in a periodic potential $V(x) = V(x+b)$.

37*. Determine the allowed energy bands of a particle moving in the periodic potential given by fig. 6. Investigate the limiting case where $V_0 \to \infty$, and $b \to 0$ while $V_0 b =$ constant.

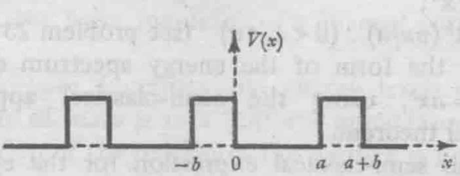

Fig. 6.

38*. A simple model of the electronic energy levels in a metal uses a one-dimensional potential of the form

$$V(x) = \frac{\hbar^2 P}{2\mu a} \sum_{n=-\infty}^{+\infty} \delta(x + na),$$

where μ is the electron mass, a the lattice constant, P a positive, dimensionless constant, and $\delta(x)$ the Dirac delta-function. Find expressions for the effective mass at the upper band edges.

39*. A particle moves in a periodic field $V(x)$:

$$V(x+a) = V(x).$$

Using a suitable semi-classical approximation obtain a transcendental equation to determine the allowed energy bands. Discuss this equation.

40. Show that for particles scattered by a complex potential, $V(x)(1+i\xi)$, the probability current density,

$$j = \frac{\hbar}{2\mu i}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right],$$

and the probability density, $\rho = \psi^*\psi$, satisfy the "continuity" equation

$$\frac{\partial j}{\partial x} + \frac{\partial\rho}{\partial t} = \frac{2V(x)\xi\rho}{\hbar}.$$

41. Use a variational principle to prove that any purely attractive one-dimensional potential has at least one bound state.

42. A particle of mass μ moves in a one-dimensional potential $\lambda V(x)$, where $V(x)$ satisfies the conditions

$$V(x) = 0, \quad x < 0; \quad V(x) = 0, \quad x > a, \quad \lambda\int_0^a V(x)\,dx < 0.$$

Prove that, if λ is sufficiently small, there exists a bound state with an energy E which is approximately given by

$$E = -\frac{\mu\lambda^2}{2\hbar^2}\left[\int_0^a V(x)\,dx\right]^2.$$

Tunnel effect

1. In studying the emission of electrons from metals, it is necessary to take into account the fact that electrons with an energy sufficient to leave the metal may be reflected at the metal surface. Consider a one-dimensional model with a potential V which is equal to $-V_0$ for $x<0$ (inside the metal) and equal to zero for $x>0$ (outside the metal) (see fig. 7), and determine the reflection coefficient at the metal surface for an electron with energy $E>0$.

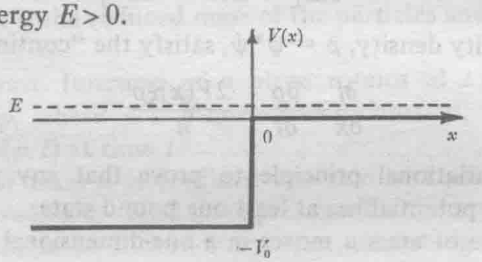

Fig. 7.

2*. In the preceding problem it was assumed that the potential changed discontinuously at the metal surface. In a real metal this change in potential takes place continuously over a region of the dimensions of the order of the interatomic distance in the metal. Approximate the potential near the metal surface by the function

$$V = -\frac{V_0}{e^{x/a}+1} \quad \text{(see fig. 8)}$$

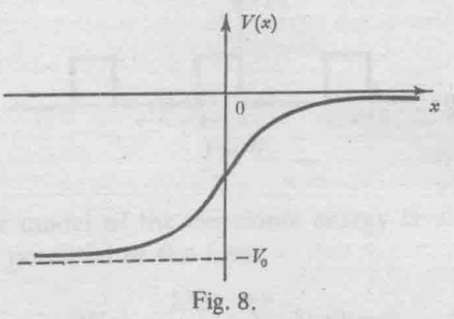

Fig. 8.

and determine the reflection coefficient of an electron with energy $E>0$.

3. Determine the coefficient of transmission of a particle through a rectangular barrier (see fig. 9).

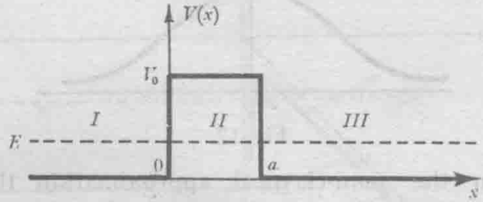

Fig. 9.

4. Determine the coefficient of reflection of a particle by a rectangular barrier in the case where $E > V_0$ (reflection above the barrier).

5. A particle is moving along the x-axis. Find the probability for transmission of the particle through a delta-function potential barrier at the origin.

6. Determine approximately the energy levels and wave functions of a particle in the symmetrical potential given by fig. 10 for the case where $E \ll V_0$ and the penetrability of the barrier is small $[(2\mu V_0/\hbar^2)\, b^2 \gg 1]$.

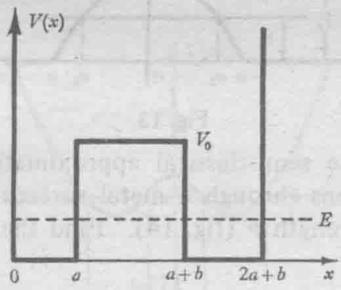

Fig. 10.

7*. Find the coefficient of transmission of a particle through a triangular barrier (see fig. 11). Consider the limiting cases of small and of large pentrability.

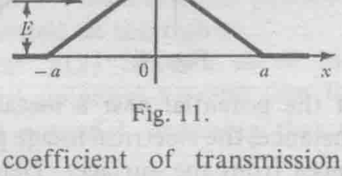

Fig. 11.

8*. Calculate the coefficient of transmission through a potential barrier $V(x) = V_0/\cosh^2(x/a)$ (see fig. 12) for particles moving with an energy $E < V_0$.

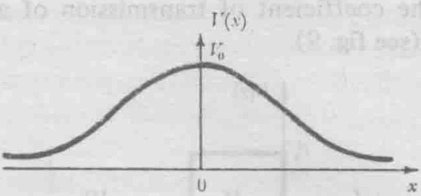

Fig. 12.

9. Evaluate in the semi-classical approximation the transmission coefficient for a parabolic potential barrier of the following form (see fig. 13):

$$V(x) = \begin{cases} V_0\left(1 - \dfrac{x^2}{a^2}\right) & \text{for } -a \leqslant x \leqslant a, \\ 0 & \text{for } |x| \geqslant a. \end{cases} \tag{1}$$

Give the criterion for the applicability of the result obtained.

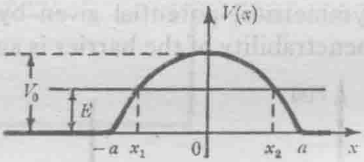

Fig. 13.

10. Calculate in the semi-classical approximation the coefficient of transmission of electrons through a metal surface under the action of a large electrical field strength F (fig. 14). Find the limits of applicability of the calculation.

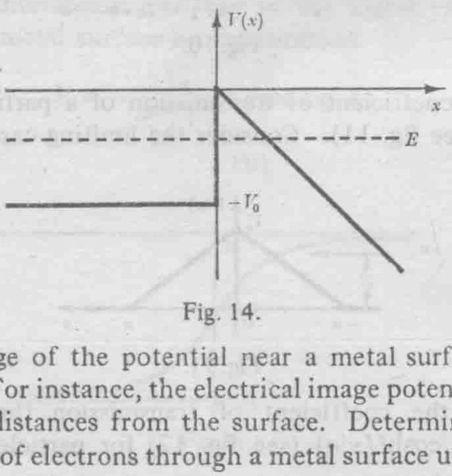

Fig. 14.

11*. The change of the potential near a metal surface is in reality a continuous one. For instance, the electrical image potential $V_{\mathrm{e\,im}} = -e/4x$ will act at large distances from the surface. Determine the coefficient of transmission D of electrons through a metal surface under the action of an electrical field, taking into account the electrical image force (fig. 15).

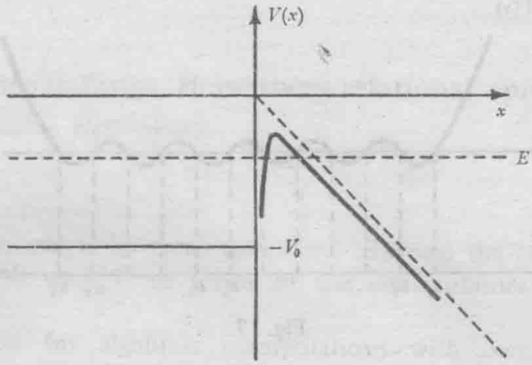

Fig. 15.

12*. A symmetrical potential $V(x)$ consists of two potential wells separated by a barrier (see fig. 16). Assuming that one may use a semi-classical argument, determine the energy levels of a particle in the potential $V(x)$. Compare the energy spectrum obtained with the energy spectrum of a single well.

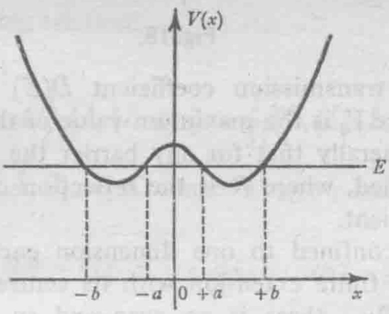

Fig. 16.

13. Assume that at $t = 0$ there exists an impenetrable partition between the two symmetrical potential wells (see preceding problem) and that a particle is in a stationary state in the well on the left.

Determine the time τ it takes after the partition is removed before the particle will be in the well on the right.

14*. The potential $V(x)$ consists of N identical potential wells separated by identical potential barriers (see fig. 17). Determine the energy levels in this potential, assuming that one can use the semi-classical approach.

Compare the energy spectrum obtained with the energy spectrum of a single well.