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REAL ANALYSIS

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Preface

This book is the outgrowth of a course at Stanford entitled Theory of Functions of a Real Variable which I have given from time to time during the last ten years. It was designed for first-year graduate students in mathematics and statistics. It presupposes not only a general background in undergraduate mathematics but also specific acquaintance with the material in an undergraduate course in the fundamental concepts of analysis. I have attempted to cover the basic material that every graduate student should know in the classical theory of functions of a real variable and in measure and integration theory, as well as some of the more important and elementary topics in general topology and in normed linear space theory. The treatment of material given here is quite standard in graduate courses of this sort, although Lebesgue measure and Lebesgue integration are done in this book before the general theory of measure and integration. I have found this a happy pedagogical practice, the student first becoming familiar with an important concrete case and then seeing that much of what he has learned can be applied in very general situations.

There is considerable independence among chapters, the chart on page xii giving the essential dependencies. The instructor thus has considerable freedom in arranging the material here into a course according to his taste. Sections which are peripheral to the principal line of argument have been starred. The Prologue to the Student lists some of the notations and conventions and makes some suggestions.

I wish to acknowledge here my indebtedness for helpful suggestions and criticism from numerous students and colleagues. Of the former I should like to mention in particular Peter Loeb, who read the manuscript and whose helpful suggestions improved the clarity of a number of arguments. Of my colleagues particular thanks are due to Herman Rubin, who provided counter examples to many of the theorems the first time I taught the course, and to John Kelley, who read the manuscript, giving helpful advice and making me omit my polemical remarks. (A few have reappeared as footnotes, however.) Finally, my thanks go to Margaret Cline for her patience and skill in transforming illegible copy into a finished typescript and to the editors at Macmillan for their forbearance and encouragement during the ten years in which this book was written.

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Prologue to the Student

This book covers a portion of the material that every graduate student in mathematics must know. For want of a better name I denote the material here by real analysis, by which I mean those parts of modern mathematics which have their roots in the classical theory of functions of a real variable. These include the classical theory of functions of a real variable itself, measure and integration, point-set topology, and the theory of normed linear spaces. This book is accordingly divided into three parts. The first part contains the classical theory of functions, including the classical Banach spaces. The second is devoted to general topology and to the theory of general Banach spaces, and the third to abstract treatments of measure and integration.

Prerequisites. It is assumed that the reader already has some acquaintance with the principal theorems on continuous functions of a real variable and with Riemann integration. No formal use of this knowledge is made here, and Chapter 2 provides (formally) all of the basic theorems required. The material in Chapter 2 is, however, presented in a rather brief fashion and is intended for review and as introduction to the succeeding chapters. The reader to whom this material is not already familiar will find it difficult to follow the presentation here. We also presuppose some acquaintance with the elements of modern algebra as taught in the usual undergraduate course. The definitions and elementary properties of groups and rings are used in some of the peripheral sections, and the basic notions of linear vector spaces are used in Chapter 10. The theory of sets underlies all of the material in this book, and I have sketched in Chapter 1 some of the basic facts from set theory. Since the remainder of the book is full of applications of set theory, the student should become adept at set theoretical arguments as he progresses through the book. I recommend that he first read Chapter 1 lightly and then refer back to it as needed. The books by Halmos [5] [1] and Suppes [17] contain a more thorough treatment of set theory and can be profitably read by the student while he reads this book.

Logical notation. We shall find it convenient to use some abbreviations for logical expressions. We use '&' to mean 'and' so that 'A & B' means 'A and B,' '$\vee$' means 'or' so that '$A \vee B$' means 'A or B (or both),' $\neg$ means 'not' or 'it is not the case that,' so that '$\neg A$' means 'it is not the case that A.' Another important notion is the one that we express by the symbol '$\Rightarrow$.' It has a number of synonyms in English, so that the statement '$A \Rightarrow B$' can be expressed by saying 'if A, then B,' 'A implies B,' 'A only if B,' 'A is sufficient for B,' or 'B is necessary for A.' The statement '$A \Rightarrow B$' is equivalent to each of the statements '$(\neg A) \vee B$' and '$\neg (A \,\&\, (\neg B))$.' We also use the notation '$A \Leftrightarrow B$' to mean '$(A \Rightarrow B) \,\&\, (B \Rightarrow A)$.' English synonyms for '$A \Leftrightarrow B$' are 'A if and only if B,' 'A iff B,' 'A is equivalent to B,' and 'A is necessary and sufficient for B.'

In addition to the preceding symbols we use two further abbreviations: '(x)' to mean 'for all x' or 'for every x,' and '$(\exists x)$' to mean 'there is an x' or 'for some x.' Thus the statement $(x)(\exists y)(x < y)$ says that for every x there is a y which is larger than x. Similarly $(\exists y)(x)(x < y)$ says that there is a y which is larger than every x. Note that these two statements are different: As applied to real numbers, the first is true and the second is false.

Since saying that there is an x such that $A(x)$ means that it is not the case that for every x we have $\neg A(x)$, we see that $(\exists x)\, A(x) \Leftrightarrow \neg (x) \neg A(x)$. Similarly $(x)\, A(x) \Leftrightarrow \neg (\exists x) \neg A(x)$. This rule is often convenient when we wish to express the negation of a complex statement. Thus

$$\begin{aligned}\neg \{(x)(\exists y)(x < y)\} &\Leftrightarrow \neg (x) \quad \neg (y) \quad \neg (x < y)\\ &\Leftrightarrow (\exists x)(y) \quad \neg (x < y)\\ &\Leftrightarrow (\exists x)(y)(y \leqq x)\end{aligned}$$

[1] Numbers in brackets refer to the bibliography, p. 279.

where we have used properties of the real numbers to infer that $\neg(x < y) \Leftrightarrow (y \leqq x)$.

We will sometimes modify the standard logical notation slightly and write $(\epsilon > 0)(\cdots)$, $(\exists\, \delta > 0)(\cdots)$, and $(\exists\, x \,\varepsilon A)(\cdots)$ to mean 'for every ϵ greater than 0 $(\cdots)$,' 'there is a δ greater than 0 such that $(\cdots)$', and 'there is an x in the set A such that $(\cdots)$'. This modification shortens our expressions. For example $(\epsilon > 0)$ $(\cdots)$ would be written in standard notation $(\epsilon)\{(\epsilon > 0) \Rightarrow (\cdots)\}$.

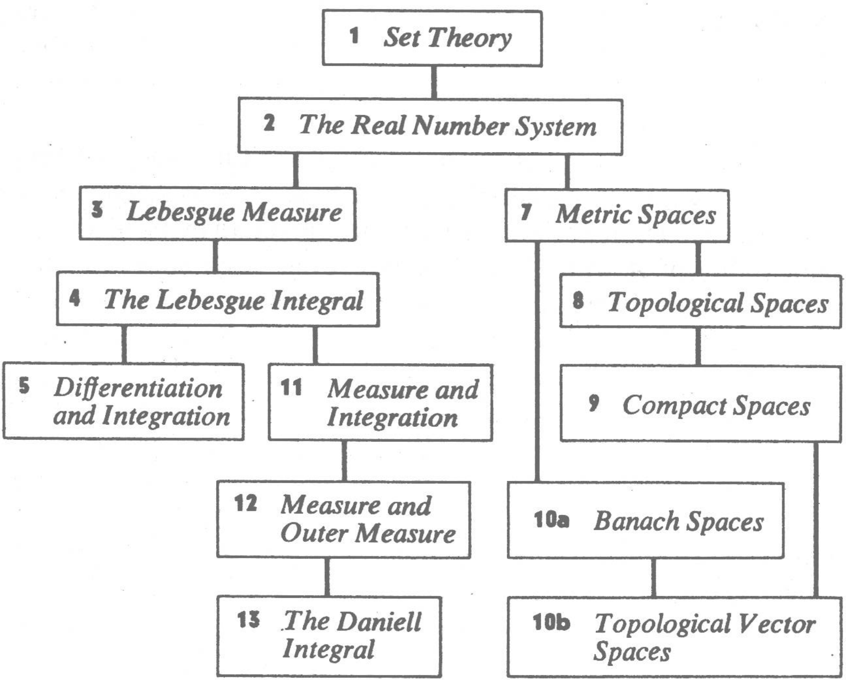
For a thorough discussion of the formal use of this logical symbolism the student should refer to Suppes [16].

Statements and their proofs. Most of the principal statements (theorems, propositions, etc.) in mathematics have the standard form 'if A, then B' or in symbols '$A \Rightarrow B$.' The *contrapositive* of $A \Rightarrow B$ is the statement $(\neg B) \Rightarrow (\neg A)$. It is readily seen that a statement and its contrapositive are equivalent, i.e. if one is true then so is the other. The direct method of proving a theorem of the form '$A \Rightarrow B$' is to start with A, deduce various consequences from it, and end with B. It is sometimes easier to prove a theorem by contraposition, i.e. by starting with $\neg B$ and deriving $\neg A$. A third method of proof is proof by contradiction or *reductio ad absurdum*. we begin with A and $\neg B$ and derive a contradiction. All graduate students should be enjoined in the strongest possible terms to eschew proofs by contradiction for two reasons: First, they are very often fallacious, the contradiction on the final page arising from an erroneous deduction on an earlier page, rather than from the incompatibility of A with $\neg B$. Secondly, even when correct such a proof sheds little, if any, insight into the connection between A and B, whereas both the direct proof and the proof by contraposition construct a chain of argument connecting A with B.

The principal statements in this book are numbered consecutively in each chapter and are variously labeled lemma, proposition, theorem, or corollary. A theorem is a statement of such importance that it should be remembered since it will be used frequently. A proposition is a statement of some interest in its own right but which has less frequent application. A lemma is usually used only for proving propositions and theorems in the same section. References to statements in the same chapter are made by giving the statement number, as Theorem 17. References to statements in another chapter take the form Proposition 3.21, meaning Proposition 21 of Chapter 3. A similar convention is followed with respect to problems. I have tried to restrict the essential use of inter-chapter references to named theorems such as 'the Lebesgue convergence theorem'; the references to numbered statements are mostly auxiliary references which the student should find it unnecessary to consult.

The proof of a theorem, proposition, etc., in this book begins with the word 'proof' and ends with the symbol '‖' which has the meaning of 'this completes the proof.' If a theorem has the form '$A \Leftrightarrow B$', the proof is usually divided into two parts, one, the 'only if' part, proving $A \Rightarrow B$, the other, the 'if' part, proving $B \Rightarrow A$.

Interdependence of the chapters. The dependence of one chapter on the various preceding ones is indicated by the following chart (except for a few peripheral references). Chapter 14 depends on most of the preceding chapters. Chapter 10a indicates Sections 1–4 and 7 of Chapter 10, Chapter 10b denotes Sections 5 and 6 of that chapter.



Contents

1 *Set Theory*

1 INTRODUCTION

One of the most important tools in modern mathematics is the theory of sets. The study of sets and their use in the foundations of mathematics was begun just before the turn of the century by Cantor, Frege, Russell, and others, and it appeared that all of mathematics could be based on set theory alone. It is in fact possible to base most of mathematics on set theory, but unfortunately this set theory is not quite as simple and natural as Frege and Russell supposed, since it was soon discovered that a free and uncritical use of set theory leads to contradictions and that set theory had to have a careful development with various devices used to exclude the contradictions. Roughly speaking, the contradictions appear when one uses sets that are "too big," such as trying to speak of a set which contains everything. In our mathematical discussions in this course we shall keep away from these contradictions by always having some set or space X fixed for a given discussion and considering only sets whose elements are elements of X, or sets (collections) whose elements are subsets of X, or sets (families) whose elements are collections of subsets of X, and so forth. In the first few chapters X will usually be the set of real numbers.

In the present chapter we shall describe some of the notions from set theory which will be useful later. Our purpose is descriptive and the arguments given are directed toward plausibility and (I hope) understanding rather than toward rigorous proof in some fixed basis for set theory. The descriptions and notations given here are for the most part consistent with the set theory described by Halmos in his book *Naive Set Theory* [5], although we assume as known, or primitive, a number of notions such as the natural numbers, the rational numbers, the notion of a function, and so forth, which can be defined (as in Halmos) in terms of the more primitive notions of set theory.

For an axiomatic treatment, I recommend Suppes' book on *Axiomatic Set Theory* [17] or the appendix to Kelley [11].

The natural numbers (positive integers) play such an important role in this book that we introduce the special symbol $\mathbf{N}$ for the set of natural numbers. We also shall take for granted the principle of mathematical induction and the well-ordering principle. The principle of mathematical induction states that if $P(n)$ is a proposition defined for each n in $\mathbf{N}$, then $\{P(1) \,\&\, [P(n) \Rightarrow P(n+1)]\} \Rightarrow (n)P(n)$. The well-ordering principle asserts that each nonempty subset of $\mathbf{N}$ has a smallest element.

The basic notions of set theory are those of set and the idea of membership in a set. We express this latter notion by ε, and write '$x \,\varepsilon\, A$' for the statement 'x is an element (or member) of A'. A set is completely determined by its members; that is, if two sets A and B have the property that $x \,\varepsilon\, A$ if and only if $x \,\varepsilon\, B$, then $A = B$. Suppose that each x in a set A is in the set B, that is, $x \,\varepsilon\, A \Rightarrow x \,\varepsilon\, B$; then we say that A is a subset of B or that A is contained in B and write $A \subset B$. Thus we always have $A \subset A$, and if $A \subset B$ and $B \subset A$ then $B = A$. It is perhaps unfortunate that the English phrase "contained in" is often used to represent both the notions ε and $\subset$, but we shall use it only in the latter context. We write '$x \not\varepsilon A$' to mean 'not $(x \,\varepsilon\, A)$,' that is, that x is not an element of A.

Since a set is determined by its elements, one of the commonest ways of determining a set is by specifying its elements as in the definition: The set A is the set of all elements x in X which have the property P. We abbreviate this by writing

$$A = \{x \,\varepsilon\, X : P(x)\},$$

and where the set X is understood we sometimes write[1]

$$A = \{x : P(x)\}.$$

We usually think of a set as having some members, but it turns out to be convenient to consider also a set which has no members. Since a set is determined by its elements, there is only one such set, and

[1] The presence (explicit or implicit) of the qualifying set X is essential. Otherwise we are confronted with the Russell paradox. Cf. Suppes [17], p. 7.