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Editors

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Proceedings of the Workshop and Symposium

ALGEBRAIC K-THEORY AND ITS APPLICATIONS

Introduction

Algebraic K-theory well illustrates the history of many rich mathematical theories. It emerged from techniques and language developed to address classical problems, in the first instance, Grothendieck's proof of his generalized Riemann Roch Theorem. It then further developed as a theoretical codification of these techniques, and in turn found new applications and mathematical connections. Algebraic K-theory at first imitated the topological K-theory of Atiyah and Hirzebruch, but the topological metaphor carried it into algebraic domains in which topology had played no historic role. It was not too surprising to see connections with algebraic geometry and with the theory of C*-algebras. But there were also dramatic and surprisingly deep connections with some of the central objects of number theory and non commutative ring theory. Moreover, in developing the theoretical apparatus of algebraic K-theory, the fundamental viewpoints and constructions of topology, particularly homotopy theory, played a central role, even in the purely algebraic and arithmetic settings.

The beginnings of the general constructions and mathematical syntheses of algebraic K-theory emerged at the Battelle Conference in 1972. In the ensuing quarter century, the subject has taken on a rich life of its own, and insinuated itself into many areas of geometry, topology, number theory and analysis. It is the setting of some deep conjectures of great consequence, and it has witnessed ongoing dramatic progress, most recently in the fundamental work of Friedlander-Suslin-Voevodsky.

The wide array of substantial mathematical domains that figure in algebraic K-theory, and the importance of its results, motivated us to organize a presentation of both its foundations, in a fashion accessible to young researchers wanting to enter or become acquainted with the field, and of the state of the art in recent research in the subject. To this end the Abdus Salam International Center of Theoretical Physics (ICTP) at Trieste hosted, in September, 1997, a two week Workshop of introductory expository lectures that surveyed most of the foundational aspects of current algebraic K-theory. These lectures were presented by some of the world's leading researchers in the field. Following this Workshop, a one week Symposium provided the venue for communication of research results from more than thirty mathematicians. These expository lectures and some of the research presentations are assembled in the present Proceedings, in the hope that they will serve as a resource to students wanting to enter the field, as a reference work on the foundations of the subject, and as a representation of some of the most recent advances in the subject.

Following is a brief survey of the scientific content of the Workshop and Symposium, which is documented in these Proceedings.

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WORKSHOP

The Workshop lectures were given under the following broad headlines - "An overview of Algebraic K-theory", K-theory and Cyclic Homology", K-theory and Algebraic Geometry" and "K-theory and Arithmetic".

1) Overview lectures

E. Friedlander and C. Weibel gave ten Overview lectures documented as Lectures I to X in these Proceedings. These include constructions of K-theory of exact, Waldhausen and symmetric monoidal categories etc., with copious examples and illustrations from algebra, number theory, algebraic topology and algebraic geometry. Fundamental results involving these constructions are also given (see Lectures I to V).

Lectures VI and VII include discussions of progress so far made on some famous conjectures, e.g. Bass finiteness conjecture, Quillen-Lichtenbaum conjectures, Beilinson conjectures.

Lecture VIII is devoted to a comprehensive discussion of Quillen and Milnor K-theory of fields including connections with Brauer group, various symbols in arithmetic, Galois and étale cohomology, Witt rings etc.

Lectures IX and X deal essentially with connections between Bloch's higher Chow groups, Quillen and Milnor K-theory of fields, Motivic and étale cohomologies. Lecture X, in particular, is an exposition of the more formal aspects of motivic cohomology - presheaves with transfer triangulated category of motives and connections with Bloch's Higher Chow groups.

2) K-theory and Cyclic Homology

(a) J.-L. Loday's contribution includes discussions of Hochschild and cyclic (co)homologies of associative algebras, Chern-Connes characters, MacLane, Lie and Leibniz (co)homology theories, and dialgebras - together with connections between them and how some of these connections do provide a basis for some aspects of "non-commutative topology".

(b) A. Connes, in his contribution, provides a lucid exposition of ideas leading to the solution of a long-standing problem of non-commutative geometry, namely, the computation of the index of transversally elliptic operators on foliations. This involves constructing for each integer 'n' a specific Hopf algebra *H(n)* and showing that it acts on the C*-algebra of the transverse frame bundle of any codimension n foliations and then doing the index computation within the cyclic cohomology of *H(n)* - a computation that was done explicitly as Gelfand Fuchs cohomology.

3) **K-theory and Algebraic Geometry**

(a) H. Gillet's contribution takes off with the construction of characteristic classes from higher K-theory of schemes X to the hyper-cohomology of X with coefficients in simplicial sheaves. He later gives a detailed account of Gersten complexes and indicates how to use K-theoretic methods to prove a conjecture of Serre that one could associate to a variety X over the complex numbers a virtual motive h(X) in the Grothendieck group of motives of smooth projective varieties.
(b) The contribution of C. Soulé, which is on K-theory and values of zeta functions, covers such topics and varieties over finite fields, K-theory of algebraic integers and Lichtenbaum's conjectures, geometry of numbers, Chern characters from K-theory to étale cohomology, Iwasawa theory and polylogarithms.

4) **K-theory and Arithmetic**

The contribution of J.-L. Colliot-Thélène includes a detailed exposition on Milnor K-theory, Galois and differential symbols, as well as Galois cohomology of discrete valuation fields.

SYMPOSIUM

The range of research articles documented in these Proceedings reflects the multidisciplinary nature of the subject. Thus we have, among others, contributions concerning: topology, involving Quillen spectral sequences in rational homotopy theory as well as various structures on higher K-theory of rings; non-commutative geometry involving cyclic cohomology and quantization in K-theoretic constructions; Algebraic Geometry involving the Chow groups of smooth varieties and the Picard groups of singular Arakelov varieties; number theory involving $K_2(Q)$ and K-theory Galois module structures; non-commutative algebra involving the Whitehead groups of division algebras; algebraic groups involving the Brauer group of quaternion algebras etc.

Acknowledgements

It is our great pleasure to thank all those who have contributed to the success of the meeting and also towards the appearance of these Proceedings.
Special thanks go to ICTP for financial support, excellent facilities and the proverbial efficiency of its staff. We express our gratitude in

particular to the mathematics staff, especially Alessandra Bergamo for her immense contributions not only towards the success of the Workshop/Symposium but also towards making the publication of these Proceedings a reality. In this connection, we also express our appreciation to Dilys Grilli (of the ICTP Publications Office) for her painstaking effort to reformat quite a number of the submissions according to World Scientific's specifications.

Finally, we thank the European Commission for supporting young European participants at the School.

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AN OVERVIEW OF ALGEBRAIC K-THEORY

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LECTURE I – The Functors K_0 and K_1

E.M. Friedlander

During the first three days of September, 1997, I had the privilege of giving a series of five lectures at the beginning of the "School on Algebraic K-Theory and Applications" at the International Centre for Theoretical Physics in Trieste. What follows are the written notes of my lectures, essentially in the form distributed to the audience. I am especially grateful to Professor Aderemi Kuku for the opportunity to participate in this workshop and whose encouragement motivated me to prepare these informal notes.

Perhaps the first new concept that arises in the study of K-theory, and one which recurs frequently, is that of the group completion of an abelian monoid. The basic example: the group completion of the monoid $\mathbf{N}$ of natural numbers is the group $\mathbf{Z}$ of integers. Recall that an abelian monoid M is a set together with a binary, associative, commutative operation $+ : M \times M \to M$ and a distinguished element $0 \in M$ which serves as an identity (i.e., $0 + m = m$ for all $m \in M$). Then we define the group completion $\gamma : M \to M^+$ by setting M^+ equal to the quotient of the free abelian group with generators $[m], m \in M$ modulo the subgroup generated by elements of the form $[m] + [n] - [m+n]$ and define $\gamma : M \to M^+$ by sending $m \in M$ to $[m]$. We frequently refer to M^+ as the *Grothendieck group* of M.

Universal property. *Let M be an abelian monoid and $\gamma : M \to M^+$ denote its group completion. Then for any homomorphism $\phi : M \to A$ from M to a group A, there exists a unique homomorphism $\phi^+ : M^+ \to A$ such that $\phi^+ \circ \gamma = \phi : M \to A$.*

This leads almost immediately to K-theory. Let R be a ring (always assumed associative with unit, but not necessarily commutative). Recall that an (always assumed left) R-module P is said to be projective if there exists another R-module Q such that $P \oplus Q$ is a free R-module.

Definition I.1. *Let $\mathcal{P}(R)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of finitely generated projective R-modules. Then we define $K_0(R)$ to be $\mathcal{P}(R)^+$.*

Warning: The group completion map $\gamma : \mathcal{P}(R) \to K_0(R)$ is frequently not injective.

Exercise: Verify that if $j : R \to S$ is a ring homomorphism and if P is a finitely generated projective R-module, then $S \otimes_R P$ is a finitely generated projective S-module. Using the universal property of the Grothendieck group, you should also check that this construction determines $j_* : K_0(R) \to K_0(S)$. Indeed, we see that $K_0(-)$ is a (covariant) functor from rings to abelian groups.

Example. If $R = F$ is a field, then a finitely generated F-module is just a finite dimensional F-vector space. Two such vector spaces are isomorphic if and only if they have the same dimension. Thus, $\mathcal{P}(F) \simeq \mathbf{N}$ and $K_0(F) = \mathbf{Z}$.

Example. Let $K/\mathbf{Q}$ be a finite field extension of the rational numbers (K is said to be a *number field*) and let $\mathcal{O}_K \subset K$ be the ring of algebraic integers in K. Thus, $\mathcal{O}$ is the subring of those elements $\alpha \in K$ which satisfy a monic polynomial $p_\alpha(x) \in \mathbf{Z}[x]$. Recall that $\mathcal{O}_K$ is a Dedekind domain. The theory of Dedekind domains permits us to conclude that

$$K_0(\mathcal{O}_K) = \mathbf{Z} \oplus Cl(K)$$

where $Cl(K)$ is the ideal class group of K (see Bass, [3]).

A well known theorem of Minkowski asserts that $Cl(K)$ is finite for any number field K (cf. [15]). Computing class groups is devilishly difficult. We do know that there are only finitely many cyclotomic fields (i.e., of the form $\mathbf{Q}(\zeta_n)$ obtained by adjoining a primitive n-th root of unity to $\mathbf{Q}$) with class group $\{1\}$. The smallest n with non-trivial class group is $n = 23$ for which $Cl(Q(\zeta_{23})) = \mathbf{Z}/3$. A check of tables shows, for example, that $Cl(\mathbf{Q}(\zeta_{100})) = \mathbf{Z}/65$.

The K-theory of integral group rings has several important applications in topology. For a group π, the integral group ring $\mathbf{Z}[\pi]$ is defined to be the ring whose underlying abelian group is the free group on the set $[g], g \in \pi$ and whose ring structure is defined by setting $[g] \cdot [h] = [g \cdot h]$. Thus, if π is not abelian, then $\mathbf{Z}[\pi]$ is not a commutative ring.

Application. *Let X be a path-connected space with the homotopy type of a C.W. complex and with fundamental group π. Suppose that X is a retract of a finite C.W. complex. Then the Wall finiteness obstruction is an element of $K_0(\mathbf{Z}[\pi])$ which vanishes if and only if X is homotopy equivalent to a finite C.W. complex.*

We now consider topological K-theory for a topological space X. This is also constructed as a Grothendieck group and is typically easier to compute than algebraic K-theory of a ring R. Moreover, results first proved for topological K-theory have both motivated and helped to prove important theorems in algebraic K-theory.

Definition I.2. *Let $\mathbf{F}$ denote either the real numbers $\mathbf{R}$ or the complex numbers $\mathbf{C}$. An $\mathbf{F}$-vector bundle on a topological space X is a continuous open surjective map $p : E \to X$ satisfying* (a.) *For all $x \in X$, $p^{-1}(x)$ is a finite dimensional* **F***-vector space.*

(b.) *There are continuous maps $E \times E \to E, \mathbf{F} \times E \to E$ which provide the vector space structure on $p^{-1}(x)$, all $x \in X$.*

(c.) *For all $x \in X$, there exists an open neighborhood $U_x \subset X$, an* **F***-vector space V, and a homeomorphism $\psi_x : V \times U_x \to p^{-1}(U_x)$ over U_x (i.e., $pr_2 = p \circ \psi_x : V \times U_x \to U_x$) compatible with the structure in (b.).*

Examples. Let $X = S^1$, the circle. The projection of the Möbius band M to its equator $p : M \to S^1$ is a rank 1, real vector bundle over S^1. Let $X = S^2$, the 2-sphere. Then the projection $p : T_{S^2} \to S^2$ of the tangent bundle is a non-trivial vector bundle. Let $X = S^2$, but now view X as the complex projective line, so that points of X can be viewed as complex lines through the origin in $\mathbf{C}^2$ (i.e., complex subspaces of $\mathbf{C}^2$ of dimension 1). Then there is a natural rank 1, complex line bundle $E \to X$ whose fibre above $x \in X$ is the complex line parametrized by x; if $E - o(X) \to X$ denotes the result of removing the origin of each fibre, then we can identify $E - o(X)$ with $\mathbf{C}^2 - \{0\}$.

Definition I.3. *Let $Vect_{\mathbf{F}}(X)$ denote the abelian monoid (with respect to $\oplus$) of isomorphism classes of* **F***-vector bundles of X. We define*

$$K^0_{top}(X) = Vect_{\mathbf{C}}(X)^+, \quad KO^0_{top}(X) = Vect_{\mathbf{R}}(X)^+.$$

(This definition will agree with our more sophisticated definition of topological K-theory given in Definition III.2 provided that the X has the homotopy type of a finite dimensional C.W. complex.)

The reason we use a superscript 0 rather than a subscript 0 for topological K-theory is that it determines a contravariant functor. Namely, if $f : X \to Y$ is a continuous map of topological spaces and if $p : E \to Y$ is an **F**-vector bundle on Y, then $pr_2 : E \times_Y X \to X$ is an **F**-vector bundle on X. This determines

$$f^* : K^0_{top}(Y) \to K^0_{top}(X).$$

Note. $K^0_{top}(X)$ is also denoted $KU(X)$ elsewhere in these notes.

Example. Let n_{S^2} denote the "trivial" rank n, real vector bundle over S^2 (i.e., $pr_2 : \mathbf{R}^n \times S^2 \to S^2$). Then $T_{S^2} \oplus 1_{S^2} \simeq 3_{S^2}$. We conclude that $Vect_{\mathbf{R}}(S^2) \to KO^0_{top}(S^2)$ is not 1-1.

Here is an early theorem of Richard Swan relating algebraic and topological K-theory. You can find a full proof, for example, in [15].

Swan's Theorem. *Let* $\mathbf{F} = \mathbf{R}$ *(respectively,* $= \mathbf{C}$*); let X be a compact Hausdorff space, and let $C(X, \mathbf{F})$ denote the ring of continuous functions $X \to$*

F. *For any* $E \in Vect_{\mathbf{F}}(X)$, *define the* $\mathbf{F}$*-vector space of global sections* $\Gamma(X,E)$ *to be*

$$\Gamma(X,E) = \{s : X \to E \ \ continuous; \ p \circ s = id_X\}.$$

Then sending E *to* $\Gamma(X,E)$ *determines isomorphisms*

$$KO^0_{top}(X) \to K_0(\mathcal{C}(X,\mathbf{R})), \quad K^0_{top}(X) \to K_0(\mathcal{C}(X,\mathbf{C})).$$

So far, we have only considered degree 0 algebraic and topological K-theory. Before we consider $K_n(R), n \in \mathbf{N}, K^n_{top}(X), n \in \mathbf{Z}$, we look explicitly at $K_1(R)$.

Denote by $GL_n(R)$ the group of invertible $n \times n$ matrices in R (i.e., an element of $GL_n(R)$ is an $n \times n$ matrix with entries in R and which admits a two-sided inverse under matrix multiplication). Denote by $GL(R)$ the union over n of $GL_n(R)$, where the inclusion $GL_n(R) \subset GL_{n+1}(R)$ sends an $n \times n$ matrix $(a_{i,j})$ to the $(n+1) \times (n+1)$ matrix whose (i,j)-th entry equals $a_{i,j}$ if both i,j are $\leq n$, whose $(n+1,n+1)$-entry is 1, and whose other entries are 0.

Definition I.4. *For any ring* R, *we define* $K_1(R)$ *by*

$$K_1(R) = GL(R)/[GL(R), GL(R)],$$

where $[GL(R), GL(R)]$ *denotes the commutator subgroup of* $GL(R)$.

Thus, $K_1(R)$ is the maximal abelian quotient group of $GL(R)$.

The following plays an important role in further developments in algebraic K-theory.

Whitehead Lemma [3]. $[GL(R), GL(R)] \subset GL(R)$ *is the normal subgroup generated by elementary matrices (i.e., those matrices with at most one non-zero diagonal element and with diagonal elements all equal to 1).*

Example. If R is a commutative ring, then sending an invertible $n \times n$ matrix to its determinant determines a well defined surjective homomorphism

$$det : K_1(R) \to R^* = \{\text{invertible elements in } \ R\}.$$

The kernel of *det* is denoted $SK_1(R)$. If $R = \mathbf{C}[x_0, x_1, x_2]/\langle x_1 + x_2 + x_3 - 1\rangle$, then $SK_1(R) = \mathbf{Z}$.

The following theorem is not at all easy, but it does tell us that nothing surprising happens for rings of integers in number fields.

Theorem of Bass-Milnor-Serre [5]. *If* $\mathcal{O}_K$ *is the ring of integers in a number field* K, *then* $SK_1(\mathcal{O}_K) = 0$.

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