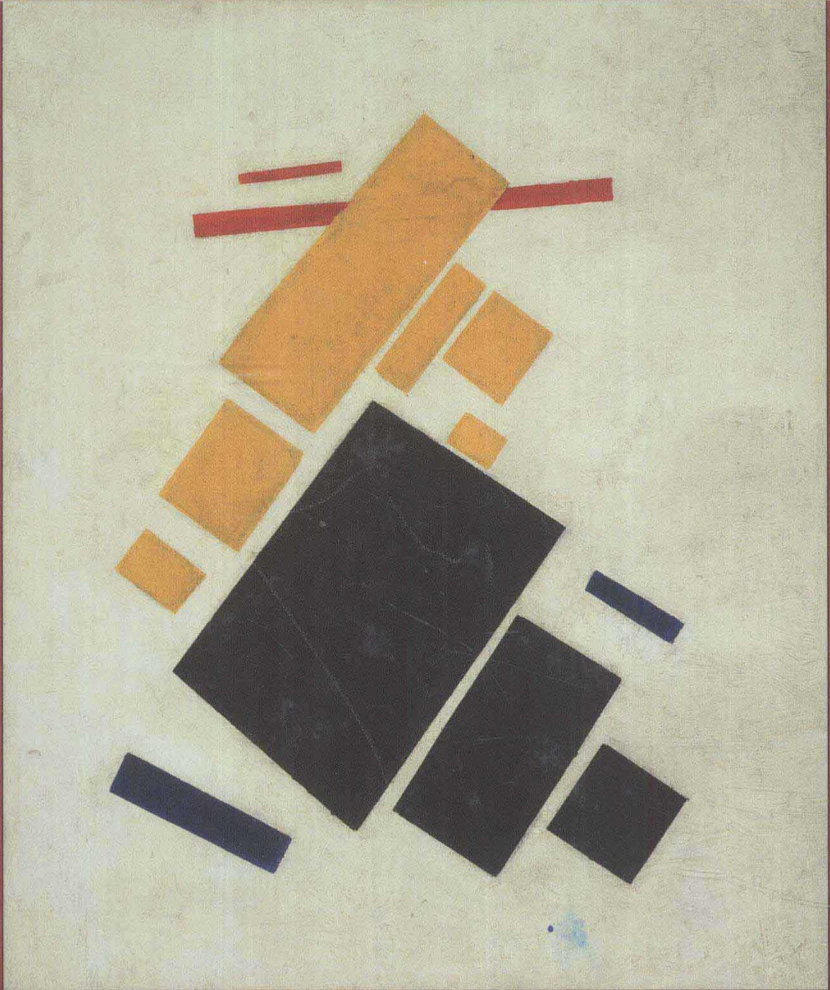
EDOUARD BRÉZIN

Introduction to Statistical Field Theory



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INTRODUCTION TO STATISTICAL FIELD THEORY

Knowledge of the renormalization group and field theory is a key part of physics, and is essential in condensed matter and particle physics. Written for advanced undergraduate and beginning graduate students, this textbook provides a concise introduction to this subject.

The textbook deals directly with the loop expansion of the free energy, also known as the background field method. This is a powerful method, especially when dealing with symmetries and statistical mechanics. In focusing on free energy, the author avoids long developments on field theory techniques. The necessity of renormalization then follows.

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Preface

These lecture notes do not attempt to cover the subject in its full extent. There are several excellent books that go much deeper into renormalization theory, or into the physical applications to critical phenomena and related topics. In writing these notes I did not mean either to cover the more recent and exciting aspects of the subject, such as quantum criticality, two-dimensional conformal invariance, disordered systems, condensed matter applications of the AdS/CFT duality borrowed from string theory, and so on.

A knowledge of the renormalization group and of field theory remains a necessary part of today's physics education. These notes are simply an introduction to the subject. They are based on actual lectures, which I gave at Sun Yat-sen University in Guangzhou in the fall of 2008. In order not to scare the students, I felt that a short text was a better introduction. There are even several parts that can be dropped by a hasty reader, such as GKS inequalities or high-temperature series. However, high-T series lead to an easy way of connecting geometrical criticality, such as self-avoiding walks and polymers or percolation to physics. I have chosen not to use Feynman diagrams; not that I think that they are unnecessary, I have used them for ever. But since I did not want to require a prior exposition to quantum field theory, I would have had to deal with a long detour, going through connected diagrams, one-particle irreducibility, and so on. I have chosen instead to base everything on the loop expansion of the free energy, not going here beyond one loop. This method, known nowadays as the background field method, is powerful (specially when dealing with symmetries) and natural from the viewpoint of statistical mechanics. (However, the more technical Chapter 13, on the renormalization of the non-linear sigma model, is aimed at readers who have some familiarity with diagrams.)

In spite of the briefness of these notes I wanted to make it clear why, after K. Wilson's work, not only were critical phenomena understood but the understanding of the meaning of renormalizability in quantum field theory changed drastically. It became clear that a beautiful renormalizable theory, such as quantum electrodynamics, was merely an *effective* theory, rather than a theory able to describe electromagnetism from astronomical distances down to vanishingly small length scales. This does not deprive QED from its exceptional beauty and its astounding agreement with experiment. In the post-Wilson analysis its renormalizability results from the fact that, like critical phenomena, the present day experiments, even at the highest presently available accelerator energies, deal with very large length scales in comparison to those at which new physics must occur. Why 'must'? It is because, unlike QCD, QED lacks 'asymptotic freedom', with the consequence that QED is 'trivial', meaning that it is only for a vanishingly small charge of the electron that it could deal with the smallest length scales. So viewing such theories, in the light of critical phenomena, told us that there has to be new physics at short distance.

Many books overlap part or most of the material of these lectures; among a long list, here is a short selection:

- John Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge: Cambridge University Press, 1996).
- Giorgio Parisi, *Statistical Field Theory* (New York: Addison-Wesley, 1988).
- J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd edn (Oxford: Oxford University Press, 2002).
- J. Zinn-Justin, *Phase Transitions and the Renormalization Group* (Oxford: Oxford University Press, 2007).
- D. J. Amit and V. Martin-Mayor, *Field Theory, the Renormalization Group and Critical Phenomena* (Singapore: World Scientific, 2005).
- C. Itzykson and J. M. Drouffe, *Statistical Field Theory*, vols 1 and 2. (Cambridge: Cambridge University Press, 1989).

Note also the historical article based on K. Wilson's lectures in Princeton (1971–72): K. G. Wilson and J. Kogut, The renormalization group, *Phys. Rep.*, **12c** (1974) 75. Several books in the long series devoted to *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. Green, and later by C. Domb and J. Lebowitz, will provide additional light on some of the topics of these lectures; see, e.g., vol. 6 of the series.

Contents

1

A few well-known basic results

This chapter is just a reminder of some basic results concerning equilibrium statistical mechanics and of a few algebraic techniques used in this book.

1.1 The Boltzmann law

For a system at equilibrium in contact with a heat bath (or thermostat) at temperature T, the configurations of the particles and the total energy are random variables. The equilibrium probability distribution for N identical particles confined in a box of volume V, whose dynamics are governed by a Hamiltonian H, is given by the Boltzmann–Gibbs distribution

$$\rho = \frac{1}{Z} \mathrm{e}^{-\beta H}, \tag{1.1}$$

in which β is related to the temperature by

$$\beta = \frac{1}{kT}. \tag{1.2}$$

1.1.1 The classical canonical ensemble

For classical particles, in three dimensions, ρ is a probability measure in the $6N$-dimensional phase space (p_a, q_a), $a = 1 \ldots 3N$ and the expectation value of an observable $A(p, q)$ is given by

$$\langle A \rangle = \int \mathrm{d}\tau \, A(p, q) \rho(p, q), \tag{1.3}$$

in which $\mathrm{d}\tau$ is the measure $\mathrm{d}\tau = \frac{1}{h^{3N}} \frac{1}{N!} \prod_1^{3N} \mathrm{d}p_a \mathrm{d}q_a$. The integrals over the positions q_a are such that every particle is confined in a box of volume V.

The factor $1/h^{3N}$, in which h has the dimension of an action (i.e., ML^2T^{-1}), makes $\mathrm{d}\tau$ dimensionless. Any constant with that dimension would work but the classical limit of quantum statistical mechanics provides Planck's constant, $h = 2\pi\hbar$.

The factor $1/N!$ is also of quantum origin: Pauli's principle allows only for one-dimensional representations of the permutation group of N particles, completely symmetric (bosons) or completely antisymmetric (fermions). This selects only one state out of the degenerate $N!$ states obtained by permutations of one of them.

The normalization is fixed by $\langle 1 \rangle = 1$, which gives the partition function Z:

$$Z(\beta, N, V) = \int d\tau e^{-\beta H}. \tag{1.4}$$

1.1.2 The quantum canonical ensemble

The density matrix ρ, given by (1.1), is an operator in the Hilbert space of symmetric states for integer spin particles, or antisymmetric states for half-integer spins, for N particles confined in a box of volume V. The expectation value of an observable A is given by

$$\langle A \rangle = \mathrm{Tr}(\rho A) = \frac{1}{Z}\mathrm{Tr} A e^{-\beta H} \tag{1.5}$$

and thus the partition function is given by

$$Z(\beta, N, V) = \mathrm{Tr}\, e^{-\beta H}. \tag{1.6}$$

If the eigenvalues of the N-body Hamiltonian are labelled as E_i, then

$$Z = \sum_i e^{-\beta E_i}. \tag{1.7}$$

If the energy E_i has a degeneracy w_i then

$$Z = \sum{}' e^{-\beta(E_i - T S_i)}, \tag{1.8}$$

in which $S_i = k \log w_i$ and the last sum runs over distinct energies. This expression shows that the dominant contributions are those that minimize the combination $E - TS$, a competition between energy and entropy to which we shall return in the next section.

Exercise 1

Quantum effects arise when the typical de Broglie wavelength associated with a particle becomes comparable to the interparticle distance. Estimate the temperature below which quantum effects should be taken into account for a gas of nitrogen of atmospheric density.

1.1.3 The grand canonical ensemble

If the system, in contact with a heat bath, can also exchange particles with a reservoir at temperature T and chemical potential μ, the number of particles is also a random variable. In the simple case in which the Hamiltonian H_N does not change the number of particles, the probability distribution is given by a collection of ρ_N given by

$$\rho_N = \frac{1}{Z_G} e^{\alpha N - \beta H_N}, \tag{1.9}$$

with

$$\mu = \frac{\alpha}{\beta}, \tag{1.10}$$

normalized by

$$Z_G(\alpha, \beta, V) = \sum_N e^{\alpha N} \mathrm{Tr}\, e^{-\beta H_N},$$

in which V is the volume of the box in which the particles are confined. (If the Hamiltonian does not conserve the number of particles, it is necessary to use the Fock space; this will not be needed within these lectures.)

1.2 Thermodynamics from statistical physics

The canonical free energy is given by

$$F(\beta, N, V) = -\frac{1}{\beta} \log Z. \tag{1.11}$$

Exercise 2

Show that the pressure, the entropy and the chemical potential of the system can all be related to the partition function. Compute the partition function for a classical gas of non-interacting particles.

1.2.1 The thermodynamic limit

The *thermodynamic limit* is the limit in which N and V go to infinity with a fixed ratio $\nu = N/V$. In this limit one can show that, for particles with short-range interactions, the canonical $\log Z$, and thus F are extensive, namely

$$\lim_{N\to\infty, V\to\infty} \bigg|_{N/V=\nu} \left(\frac{1}{N} \log Z \right) \tag{1.12}$$

exists and is a function of the two intensive variables ν and β. Similarly, for the grand canonical ensemble, $\lim_{V\to\infty} 1/V \log Z_G$ exists and is a function of the intensive variables, temperature and chemical potential.

Exercise 3

Verify this extensivity for N free classical particles in a box. Reminder: Stirling's formula $N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N\left(1+O\left(\frac{1}{N}\right)\right)$.

For charged particles, such as electrons with Coulomb interactions, the thermodynamic limit exists, provided that (a) the system is neutral, i.e., the charge of the ions compensates the charge of the electrons, (b) the system is quantum mechanical, (c) Pauli's principle is taken into account.[1]

Exercise 4

Assume that the potential energy of N interacting classical particles is a homogeneous function

$$V(\lambda q_1 \cdots \lambda q_{3N}) = |\lambda|^s V(q_1 \cdots q_{3N}).$$

Show that the pressure $p(\nu, T)$*, where* $\nu = N/V$*, satisfies the relation*

$$p(\nu, T) = T^{1-3/s}\varphi(\nu T^{3/s}).$$

Assume that at low temperature T_0 *the isotherm in the* (p, V) *plane presents a phase transition between two phases of different densities. Can there be a critical point for this phase transition, i.e., a temperature at which the transition between the two phases disappears?*

1.3 Gaussian integrals and Wick's theorem

1. *One variable*

$$\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}. \tag{1.13}$$

2. *n variables*

$$\int_{R^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}. \tag{1.14}$$

$A = A^t$ is here a real symmetric matrix with positive eigenvalues. It can thus be diagonalized by an orthogonal transformation ω, i.e., $A = \omega^t D\omega$, in which D is the diagonal matrix of the eigenvalues $(a_1, \ldots, a_n)$ of A. The change of variables $\omega x = y$ whose Jacobian $(|\det\omega|^{-1})$ is equal to one, leads to the solution.

3. *n variables in a source*

$$\frac{\int_{R^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j + \sum b_i x_i}}{\int_{R^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j}} = \mathrm{e}^{\frac{1}{2}\sum b_i A^{-1}_{ij} b_j}. \tag{1.15}$$

Translate $x = y + A^{-1}b$.

[1] J. Lebowitz and E. Lieb, *Phys. Rev. Lett.*, **22** (1969) 631.

4. *Wick's theorem*
   Apply to (1.15) the operation $\frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_{2n}}}$ and then set all the $b_i = 0$. The l.h.s. gives

$$\langle x_{i_1} \cdots x_{i_{2n}} \rangle = \frac{\int_{R^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j} x_{i_1} \cdots x_{i_{2n}}}{\int_{R^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j}}. \tag{1.16}$$

Applying this to the r.h.s. of (1.15) we can limit ourselves to the term $\frac{1}{(n)!2^n} \times \left(\sum b_i A^{-1}_{ij} b_j\right)^n$; indeed terms of lower degree in the expansion of the exponential will give zero by differentiation; terms of higher degree will give zero because they are left with b and vanish at $b = 0$. Therefore,

$$\langle x_{i_1} \cdots x_{i_{2n}} \rangle = \frac{\partial}{\partial b_{i_1}} \cdots \frac{\partial}{\partial b_{i_{2n}}} \frac{1}{(n)!2^n} \left(\sum b_i A^{-1}_{ij} b_j\right)^n. \tag{1.17}$$

Define a complete pairing of the $\frac{\partial}{\partial b}$ such that each $\frac{\partial}{\partial b_i}$ has a partner. For this particular pairing, the two paired differentiations go to the same $\sum$, but there are $n!$ ways of associating the sums and the chosen pairing. Once this association is made, one has simply to note that

$$\frac{\partial}{\partial b_k} \frac{\partial}{\partial b_l} \sum b_i A^{-1}_{ij} b_j = 2A^{-1}_{kl}.$$

Therefore the $n!2^n$ cancels and we are left with the result, known as Wick's theorem, for Gaussian integrals.

$$\langle x_{i_1} \cdots x_{i_{2n}} \rangle = \sum_{\text{pairings}} \prod_{\substack{\text{each} \\ \text{pair}}} A^{-1}_{i_a i_b}. \tag{1.18}$$

Exercise

Compute the integral

$$I = \frac{\int_{R^2} \mathrm{d}x\mathrm{d}y\ x^4 y^2 \mathrm{e}^{-(x^2+xy+2y^2)}}{\int_{R^2} \mathrm{d}x\mathrm{d}y \mathrm{e}^{-(x^2+xy+2y^2)}}.$$

Answer

$$I = \frac{144}{343},$$

since

$$I = 3(\langle xx \rangle)^2 \langle yy \rangle + 12 \langle xx \rangle (\langle xy \rangle)^2,$$

in which

$$\langle xx \rangle = A^{-1}_{11} \quad \langle xy \rangle = A^{-1}_{12} \quad \langle yy \rangle = A^{-1}_{22},$$

with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad A^{-1} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}.$$

1.4 Functional derivatives

A functional $F\{f\}$ is an application from a space of functions f to a complex or real number F. For instance the action integral for the motion of a particle, located at the position $q(t)$ at time t, with potential energy $V(q)$ is the functional of the trajectory given by

$$S\{q\} = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{m}{2}\dot{q}^2 - V(q)\right]. \tag{1.19}$$

Let us work with functions f of a single real variable x (the generalization to functions of more variables is immediate). The derivative of the functional with respect to $f(x)$ at $x = x_0$ is defined as follows. Let us consider an increment $\epsilon\delta_\eta(x - x_0)$; the function $\delta_\eta(x)$ is centred at the origin, and it has a width η; it is normalized to one, i.e., $\int_{\mathcal{R}} \delta_\eta(x)\mathrm{d}x = 1$. When η goes to this zero, this increment approaches the Dirac distribution $\delta(x)$. (For instance $\delta_\eta(x) = \frac{1}{\eta\sqrt{2\pi}}\mathrm{e}^{-x^2/2\eta^2}$.) One computes next the increment of the functional

$$\Delta F = F\{f + \epsilon\delta_\eta(x - x_0)\} - F\{f\}. \tag{1.20}$$

The functional derivative of F at x_0 is defined as

$$\left.\frac{\delta F}{\delta f}\right|_{x_0} = \lim_{\eta\to 0}\lim_{\epsilon\to 0} \frac{\Delta F}{\epsilon}. \tag{1.21}$$

The limits have to be taken in the order indicated: if we let ϵ go to zero first, we avoid non-linearities in δ_η. In the opposite order we would encounter powers of δ_η, which do not have a limit when η goes to zero.

Let us apply this to the above action functional:

$$\begin{aligned}\lim_{\epsilon\to 0}&\frac{1}{\epsilon}[S\{q(t) + \epsilon\delta_\eta(t - t_0)\} - S\{q\}]\\ &= \int_{t_1}^{t_2} \mathrm{d}t[m\dot{q}\ \dot{\delta}_\eta(t - t_0) - V'(q)\delta_\eta(t - t_0)].\end{aligned} \tag{1.22}$$

After an integration by parts of the first term one ends up with

$$\frac{\delta S}{\delta q}(t_0) = -m\ddot{q}(t_0) - V'(q(t_0)) \tag{1.23}$$

and Newton's law is just given by the vanishing of this functional derivative for any t_0: the action is stationary (in fact a minimum) for the classical trajectory.

1.5 *d*-dimensional integrals

The rules are simple but they may surprise the reader who sees them for the first time. Whenever the dimension d is an integer, the d-dimensional integral is the ordinary integral over the whole space R^d. But, for arbitrary d, one applies the following rules:

(a) $\int d^d q f(q+p) = \int d^d q f(q)$,
(b) $\int d^d q f(\lambda q) = |\lambda|^{-d} \int d^d q f(q)$.

If q_1 is a d_1-dimensional vector and q_2 is a d_2-dimensional vector and $f(q) = g_1(q_1) g_2(q_2)$ with $d = d_1 + d_2$, then

(c) $\int d^d q f(q) = \int d^{d_1} q_1 g_1(q_1) \int d^{d_2} q_2 g_2(q_2)$.

Consequences:

- From (b) the only finite solution to an integral, such as $\int d^d q (q^2)^k$ is

$$\int d^d q (q^2)^k = 0$$

  for any positive or negative real number k, including $k = 0$. Note that this integral never exists as an ordinary integral for integer dimensions. The consistency of this rule will be checked below.
- The same would apply to any scale-invariant integral, such as

$$\int d^d q_1 d^d q_2 (q_1^2)^k \left[(q_1 + q_2)^2\right]^l = 0.$$

- From (c)

$$\int d^d q \, e^{-q^2} = \left[\int_{-\infty}^{+\infty} dx e^{-x^2}\right]^d = \pi^{d/2}.$$

Let us use these rules to calculate simple integrals:

$$\begin{aligned} \int d^d q (q^2+1)^{-k} &= \frac{1}{\Gamma(k)} \int d^d q \int_0^\infty e^{-\lambda(q^2+1)} \lambda^{k-1} d\lambda \\ &= \frac{\pi^{d/2}}{\Gamma(k)} \int_0^\infty d\lambda \lambda^{k-d/2-1} e^{-\lambda} = \frac{\pi^{d/2} \Gamma(k-d/2)}{\Gamma(k)}. \end{aligned}$$

One can also compute this integral in 'spherical' coordinates:

$$\begin{aligned} \int d^d q (q^2+1)^{-k} &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dx \, x^{d-1} \frac{1}{(x^2+1)^k} \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{2} \int_0^1 dy \, y^{k-d/2-1} (1-y)^{d/2-1} \\ &= \frac{\pi^{d/2} \Gamma(k-d/2)}{\Gamma(k)} \end{aligned}$$

(change $1/(1+x^2) = y$). It is easy to verify on examples, such as $d = 3$ and $k = 2$, that whenever the integral exists in the ordinary sense it is indeed given by this result.

To check the consistency of rule (b) let us compute

$$J = \int \mathrm{d}^d q \frac{1}{q^2(q^2+1)}.$$

If we use spherical coordinates,

$$J = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \mathrm{d}x \, x^{d-3} \frac{1}{x^2+1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{2} \int_0^1 \mathrm{d}y \, y^{1-d/2}(1-y)^{d/2-2}$$
$$= \frac{\pi^{d/2}}{\Gamma(d/2)} \Gamma(2-d/2)\Gamma(d/2-1).$$

It is easy to verify that, for $d = 3$, J exists as an ordinary integral and is indeed given by this result. Alternatively, using the identity $\frac{1}{q^2(q^2+1)} = \frac{1}{q^2} - \frac{1}{q^2+1}$, we find, from rule (b) and the above $k = 1$ result,

$$J = 0 - \pi^{d/2}\Gamma(1-d/2),$$

and it is easy to check that this coincides with the above result for J.

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