INTRODUCTION TO LINEAR ALGEBRA

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Dedicated to my parents Bernard and Mariam Mizel, to my daughter Anne-Marie, and especially to Phyllis, the "number one guinea pig."

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Preface

Linear algebra is the earlier of two mathematical disciplines devoted to the study of that broad and useful notion called linearity. The other discipline, linear analysis, involves much more sophisticated techniques and consequently occurs much later in one's career. In the present instance a study of the lesser discipline foreshadows practically every one of the conclusions of the more subtle topic. Therefore a student of linear algebra is in the fortunate position of gaining a great deal of insight into more advanced matters as he achieves a mastery of this subject.

The most significant of one's early experiences with linearity arise from two sources: the study of lines and planes in analytic geometry and the study of systems of linear algebraic equations. We assume that the first of these is reasonably well understood, at least on an intuitive basis. Unfortunately, experience shows that the second topic is known only on a primitive level by students entering a linear algebra course.

The main commitment of this book is to a geometric treatment of linear algebra. This is achieved by an early introduction of the inner product and the associated notions of length and angle. Even though the inner product is not always used, it is always available. The geometrical viewpoint is further aided by employment of coordinate-free notation in the presentation of an essentially determinant-free theory. Experience has shown us that student resistance to notions of length and angle is lower than student resistance to such items as the axioms for a vector space and the notion of subspace. For this reason the book is written in such a way that the Euclidean vector spaces R^n and their subspaces provide an adequate format for realization of most material. This is especially desirable when the text is read by a college sophomore. The instructor might be advised to begin Chap. 2 with a very light treatment of the axioms for a vector space and of the notion of subspace in order to get to the inner product as soon as possible.

It is a goal of this book to give the college sophomore a one-semester view of linear algebra which includes a comprehensive look at the spectral theorem for symmetric operators, Chaps. 1 to 5. Moreover, the additional material in Chaps. 6 to 8 challenges juniors and seniors; it also permits use of this text for a two-semester course. Chapter 1 gives a presentation of the Gauss procedure for reducing linear systems of equations and thus provides the student with his main tool for dealing with the subject manually. Chapter 5 gives attention to the spectral theorem. In order to cover this material in one semester it may be necessary to make some omissions. Dispensable sections are Sec. 3 and the end of Sec. 1 of Chap. 3, Sec. 2 of Chap. 4, and Sec. 2 of Chap. 5.

We have received generous help from our colleagues. In particular, Henry Leonard read various portions of the manuscript and made many stimulating suggestions. Walter Noll is responsible not only for several specific ideas used in the presentation (notably in the discussions of gradient and of trace) but for instilling in the authors an appreciation of his profound viewpoint toward linear algebra.

Joel Williamson was an active help in proofreading both the early drafts and the final proofs. To him and to the other students who were subjected to this experiment, our hope that their suffering was not in vain. Our appreciation goes to Mary Ellen Hanlein (nee Kay), Judy Lewis, and Ida Laquarta for typing various drafts of the manuscript.

A. D. Martin
V. J. Mizel

It is my sad duty to announce here that Allan Martin died shortly after the main draft for this book was completed. His untimely death removed from the scholastic community a fine mathematician and outstanding pedagogue. It deprived me of a close friend. *Vale, amice.*

V. J. M. June 29, 1966

To the Student *Whatever else you do, read the entire Problems and Comments after each section. Otherwise you will miss important topics not treated in the main body of the text.*

Contents

CHAPTER 1

Systems of Linear Equations, Matrices, Arrows

In one way or another linear algebra is fundamentally concerned with systems of linear equations. Although our approach to linear algebra is geometric in every way possible, most applications of the concepts of the subject involve at least one system of linear equations. For this reason much of this first chapter is devoted to informing the reader of the manual skill he will require in dealing with such systems. The rest of the chapter treats matrices and arrows, topics which anticipate the basic notions of vector space theory.

1. Linear systems and Gauss procedures

In the family of all linear systems there are those which have no solutions, those which have only a single solution and those which have many solutions. It is very important to a worker in this subject that he be able to place a given system of equations into one of these three categories. Sometimes this can be done effortlessly and sometimes it cannot. Certainly the system

$$\begin{aligned} x + y &= 1 \\ x + y &= 0 \end{aligned}$$

has no solution, since there is no pair (x, y) of numbers x and y whose sum is simultaneously 1 and 0. On the other hand, the system

$$\begin{aligned} x &= 1 \\ y &= x + 2 \end{aligned}$$

has only one solution: $(x, y) = (1, 3)$. Finally, the one-equation system

$$y + x - 1 = 0$$

has many solutions. Two of them are $(x, y) = (0, 1)$ and $(x, y) = (2, -1)$. In fact, the totality of solutions of this equation is infinite; in a familiar way it is identified with the totality of all points (x, y) in the plane of analytic geometry which lie on the line with y-intercept 1 and slope -1 (Fig. 1.1):

$$y + x - 1 = 0$$

Similarly, the preceding system (Fig. 1.2)

$$\begin{aligned} x &= 1 \\ y &= x + 2 \end{aligned}$$

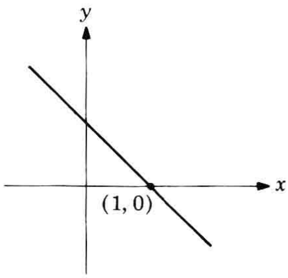

Fig. 1.1

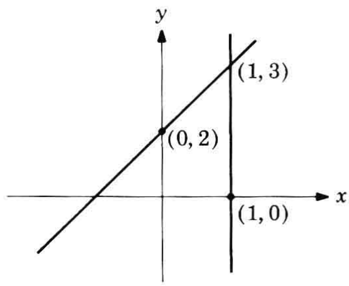

Fig. 1.2

consists of the equations of two lines which intersect in a single point. The remaining system (Fig. 1.3)

$$\begin{aligned} x + y &= 1 \\ x + y &= 0 \end{aligned}$$

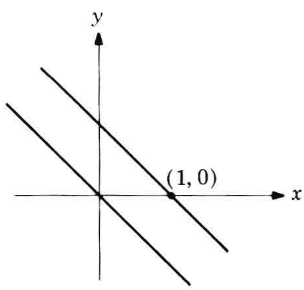

Fig. 1.3

consists of the equations of two parallel lines which do not intersect at all.

On the other hand, the system

$$\begin{aligned} x + y + z + 2w &= 2 \\ 2x + y + z + 4w &= 3 \\ -3x + 2y + z - 7w &= 1 \end{aligned} \tag{1.1}$$

does not yield its secrets as easily as do the three systems discussed above. Nevertheless, many problems in the course of their solution demand that the investigator acquire an intimate knowledge of systems which are at least as complicated as (1.1). Some of these systems have infinitely many solutions, in which case it is manifestly impossible for a human being to write them all down on a piece of paper. Others may have only a single solution, whose explicit form is then required. In the former case we say that the problem requires a descriptive solution, in the latter a numerical solution. Whether the problem requires a descriptive or a numerical solution, or indeed has no solution at all, we must be able to simplify a system of linear equations down to the bones. Therefore the next few pages are devoted to describing a method of constructing for each linear system a formula which gives all its solutions. The method is due to Gauss and Jordan. It consists of a finite sequence of steps no one of which requires the operator to guess or to make a creative decision. Accordingly it can be performed by a digital computer, and in this context it is called an *algorithm.* In order to carry out the above program it would be desirable to state the goal being pursued as well as the means to that goal. Unfortunately, it is considerably easier to describe the algorithm than to describe the sort of final formula it produces. We will eventually survey such results, but only after the course of action has been described.

A preliminary application to the very simple system

$$\begin{aligned} 2x + 4y &= 1 \\ x - y &= 2 \end{aligned} \tag{1.2}$$

should help to clarify the idea of the method as well as fix the notation we will use. We systematically modify (1.2) to arrive at a new system of equations as follows:

$$\begin{matrix} 2x + 4y = 1 \\ x - y = 2 \end{matrix} \xrightarrow{\frac{1}{2}[1]} \begin{matrix} x + 2y = \frac{1}{2} \\ x - y = 2 \end{matrix} \xrightarrow{[2] - 1[1]}$$

$$\begin{matrix} x + 2y = \frac{1}{2} \\ 0x - 3y = \frac{3}{2} \end{matrix} \xrightarrow{-\frac{1}{3}[2]} \begin{matrix} x + 2y = \frac{1}{2} \\ 0x + y = -\frac{1}{2} \end{matrix} \xrightarrow{[1] - 2[2]}$$

$$\begin{aligned} x + 0y &= \tfrac{3}{2} \\ 0x + y &= -\tfrac{1}{2} \end{aligned} \tag{1.2a}$$

In the above formalism we have introduced symbols of the sort $\xrightarrow{c[i]}$ and $\xrightarrow{[i] + c[j]}$ to represent operations used in modifying a linear system. Namely, $\xrightarrow{c[i]}$ denotes replacement of equation i by its c-multiple while $\xrightarrow{[i] + c[j]}$ denotes replacement of equation i by its sum with the c-multiple of equation j. It is understood that the system to which an operation is to be applied is always the one most recently constructed.

The reader is undoubtedly aware of ways to bypass some of the steps we have used in arriving at the final result $x = \frac{3}{2}$, $y = -\frac{1}{2}$, but in less trivial cases the systematic nature of the technique is of great help.

Now consider the system (1.1). Certainly it would be simplified if we were to multiply both sides of all three equations by zero:

$$\begin{aligned} 0x + 0y + 0z + 0w &= 0 \\ 0x + 0y + 0z + 0w &= 0 \\ 0x + 0y + 0z + 0w &= 0 \end{aligned} \tag{1.1a}$$

However, the resulting system (1.1*a*) has the property that every quadruple (x, y, z, w) of numbers, x, y, z, and w is a solution of it. The system (1.1) lacks this property as, for example, the quadruple (0, 0, 0, 0) is not one of its solutions. The transition from (1.1) to (1.1*a*) is essentially a matter of moving our attention from one problem to a second problem which has nothing to do with the first. In order to avoid this kind of irrelevancy we must pick our maneuvers with care.

(1.3) DEFINITION. **Admissibility.** An operation on a system of equations is termed *admissible* if the resulting system has exactly the same solutions as the original system. Any operation by which a solution is lost or gained is *inadmissible.*

Of course there are many admissible operations, but we shall be concerned with only two:

(M) Multiplying an equation by a nonzero number: replacing $[k]$ by its c-multiple, $c[k]$

(A) Adding a multiple of one equation to another: replacing $[j]$ by the equation $[j] + c[k]$

An equally simple third operation, the interchange of equation $[j]$ with equation $[k]$, is not included here because it is reducible to (M) and (A). See Sec. 2, PC 15 (PC = Problems and Comments).

The operations (M) and (A) are both special cases of a compound "Gauss" operation

(G) Replacing $[j]$ by the equation $b[j] + c[k]$, where $b \neq 0$

The fact that the above operations are all admissible is the underlying reason for the success of the technique we will describe. As such, it is sufficiently important to be prominently displayed.

(1.4) THEOREM. *Any operation of type* (M), (A), *or* (G) *when applied to a system of equations leaves the set of solutions unchanged: operations* (M), (A), *and* (G) *are admissible.*

Since we have as yet no way to describe a general system of equations, we cannot possibly give a proof of (1.4) here. The following remarks however contain the gist of the proof, and a complete presentation will be left to the problems in Sec. 2.

Consider the effect of applying the above operations to the following system of equations.

$$\begin{aligned} x + y + z &= 2 \\ 2x + y - z &= 3 \\ 4x + 3y + z &= 7 \end{aligned} \tag{1.5}$$

Since operations of type (M) and type (A) are both special cases of operations of type (G), we consider the effect of a typical operation of type (G) on (1.5). If we replace [2] by $b[2] + c[1]$, we get

$$\begin{aligned} x + y + z &= 2 \\ (2b + c)x + (b + c)y + (-b + c)z &= 3b + 2c \\ 4x + 3y + z &= 7 \end{aligned} \tag{1.5a}$$

Now if $(x, y, z) = (s_1, s_2, s_3)$ is a solution of (1.5), then the same numbers also form a solution of (1.5*a*): the equality $(2b + c)s_1 + (b + c)s_2 + (-b + c)s_3 = 3b + 2c$ follows from the equalities $s_1 + s_2 + s_3 = 2$ and $2s_1 + s_2 - s_3 = 3$ by laws of arithmetic. Thus the set of solutions of (1.5*a*) includes every solution of (1.5), as well as (conceivably) others.

To show that as a matter of fact (1.5*a*) has *no* solutions which are not also solutions of (1.5), so that (1.5) and (1.5*a*) have the same set of solutions, it is necessary to observe that the system (1.5) itself is constructible from (1.5*a*) by a step of type (G):

$$\text{Replacing [2] by } b^{-1}[2] - b^{-1}c[1]$$

If we now repeat the reasoning of the preceding paragraph starting this time with a typical solution of (1.5*a*), we will conclude as desired that (1.5*a*) has no solutions which are not also solutions of (1.5).

(1.6) Definition. Any finite sequence of operations of type (G) applied to a system of equations is called a **Gauss procedure.**

We may now state an important result following immediately from Theorem (1.4).

(1.7) Theorem. *Any application of a Gauss procedure to a system of equations leaves the set of solutions unchanged: a Gauss procedure is an admissible operation.*

Notice that Theorem (1.7) provides a justification for the method we used earlier to solve the system of two linear equations (1.2). Since (1.2*a*) was constructed from (1.2) by a sequence of four operations each of type (G), we are assured that the solution of (1.2*a*), given by inspection, is indeed the solution of (1.2) as well.

Now let us apply these methods to solve a somewhat more interesting system.

(1.8) Example

$$\begin{aligned} 2x + 2y + 2z + w &= -2 \\ 2x + y + 2z - w &= 1 \\ x + 2y + 4z + w &= -1 \end{aligned} \tag{L}$$

The following Gauss procedure solves this system. It is in fact an immediate extension of the technique used earlier to solve (1.2).

Stage 1. x-reduce the system: Choose an equation of (L) whose x-coefficient is not zero; multiply this equation by a factor producing the

x-coefficient unity; add a multiple of the modified equation to each of the other equations so as to produce zero x-coefficients. In the present example any equation will do, so we choose [1]. We then have

$$
\begin{array}{rcl}
2x + 2y + 2z + w = -2 & \xrightarrow{\tfrac{1}{2}[1]} & x + y + z + \tfrac{1}{2}w = -1 \\
2x + y + 2z - w = 1 & & 2x + y + 2z - w = 1 \quad [2] - 2[1] \\
x + 2y + 4z + w = -1 & & x + 2y + 4z + w = -1 \xrightarrow{[3] - 1[1]}
\end{array}
$$

$$
(\mathrm{L}_1) \qquad
\begin{array}{l}
x + y + z + \tfrac{1}{2}w = -1 \\
0x - y + 0z - 2w = 3 \\
0x + y + 3z + \tfrac{1}{2}w = 0
\end{array}
$$

Stage 2. y-reduce (L_1): Choose any equation of (L_1) whose x-coefficient is zero but whose y-coefficient is not zero; multiply this equation by a factor producing the y-coefficient unity; add a multiple of the modified equation to each of the other equations so as to produce zero y-coefficients. In this case either [2] or [3] will do, so choose [2]. We then have

$$
\begin{array}{rcl}
x + y + z + \tfrac{1}{2}w = -1 & & x + y + z + \tfrac{1}{2}w = -1 \quad [1] - 1[2] \\
0x - y + 0z - 2w = 3 & \xrightarrow{-1[2]} & 0x + y + 0z + 2w = -3 \\
0x + y + 3z + \tfrac{1}{2}w = 0 & & 0x + y + 3z + \tfrac{1}{2}w = 0 \xrightarrow{[3] - 1[2]}
\end{array}
$$

$$
(\mathrm{L}_2) \qquad
\begin{array}{l}
x + 0y + z - \tfrac{3}{2}w = 2 \\
0x + y + 0z + 2w = -3 \\
0x + 0y + 3z - \tfrac{3}{2}w = 3
\end{array}
$$

Stage 3. z-reduce (L_2): Choose an equation of (L_2) whose x- and y-coefficients are both zero but whose z-coefficient is not zero; multiply this equation by a factor producing the z-coefficient unity; add a multiple of the modified equation to each of the other equations so as to produce zero z-coefficients. In this case only [3] will do. One then has

$$
\begin{array}{rcl}
x + 0y + z - \tfrac{3}{2}w = 2 & & x + 0y + z - \tfrac{3}{2}w = 2 \quad [1] - 1[3] \\
0x + y + 0z + 2w = -3 & & 0x + y + 0z + 2w = -3 \\
0x + 0y + 3z - \tfrac{3}{2}w = 3 & \xrightarrow{\tfrac{1}{3}[3]} & 0x + 0y + z - \tfrac{1}{2}w = 1 \longrightarrow
\end{array}
$$

$$
(\mathrm{L}_3) \qquad
\begin{array}{l}
x + 0y + 0z - w = 1 \\
0x + y + 0z + 2w = -3 \\
0x + 0y + z - \tfrac{1}{2}w = 1
\end{array}
$$

Stage 4. w-reduce (L_3): Choose an equation of (L_3) whose x-, y-, and z-coefficients are all zero but whose w-coefficient is not zero; multiply this equation by a factor producing the w-coefficient unity; etc. (L_3) has no such equation and so our Gauss procedure terminates with the system (L_3), which is called a reduced form of (L).

From (L_3) we have that (x, y, z, w) is a solution if and only if

$$
(\mathrm{L}_3') \qquad
\begin{array}{l}
x = w + 1 \\
y = -2w - 3 \\
z = \tfrac{1}{2}w + 1
\end{array}
$$

This shows that w may be chosen arbitrarily, and after each such choice there is only one value for each of the quantities x, y, and z such that (x, y, z, w) is a solution of (L_3). According to Theorem (1.7) the solutions of (L_3) coincide with the solutions of (L), so the above procedure has shown that (L) has an infinite number of solutions, all of them displayed in formula (L_3').

Problems and Comments

1. Test the assertion in (1.8) that all solutions of (L_3') are solutions of (L) by explicitly checking whether the solution of (L_3') corresponding to $w = 3$ solves (L). Likewise for $w = 0$ and $w = \sqrt{2}$.

2. Use the admissible operations (M) and (A) to solve the following systems:

$$(a)\ \begin{aligned} 2x - 3y &= 1 \\ x + y &= -1 \end{aligned} \qquad (b)\ \begin{aligned} x + 3y &= 5 \\ 2x + 6y &= -1 \end{aligned}$$

$$(c)\ \begin{aligned} x - y &= 1 \\ x + y &= -1 \end{aligned}$$

3. Use the technique of Example (1.8) to solve the following systems:

$$(a)\ \begin{aligned} 2x + 3y + z &= 5 \\ x \qquad - z &= 1 \\ 2x - 9y - 11z &= -5 \end{aligned} \qquad (b)\ \begin{aligned} 2x - y + 3z - w &= 11 \\ x \qquad - z + w &= -2 \\ -2x - 2y + 2z - 3w &= 4 \end{aligned}$$

4. By improvising where necessary on the technique of Example (1.8) solve

$$\begin{aligned} 2x + 2y - z + w &= -1 \\ x + y + 2z - w &= 2 \\ 3x + 3y + z + 2w &= 0 \end{aligned}$$

5. Consider the generalization of steps of type (G) obtained by discarding the condition $b \neq 0$:

$(\bar{G})$ Replacement of $[j]$ by $b[j] + c[k]$

Determine whether Theorem (1.4) holds with this definition; give either a proof or a counterexample.

***6.** Devise a Gauss procedure which, when applied to

$$(a)\ \begin{aligned} x + 2y &= 1 \\ x - y &= 3 \end{aligned}$$

yields

$$(b)\ \begin{aligned} x - y &= 3 \\ x + 5y &= -1 \end{aligned}$$

Hint: First solve both systems by the scheme used for (1.2).

7. Give conditions on the quantities a_1, a_2, b_1, b_2, d_1, and d_2 so that

$$(L)\qquad \begin{aligned} a_1x + b_1y &= d_1 \\ a_2x + b_2y &= d_2 \end{aligned}$$

has $(x, y) = (1, 3)$ as its only solution.

* This symbol denotes moderately hard problems; ** denotes still harder ones.

2. *Gauss reduction of linear systems*

In this section we will amplify the discussion of Gauss procedures begun in Sec. 1. Up to now we have not even said what distinguishes a linear system of equations from a nonlinear system.

(2.1) DEFINITION. **Linear System.** By a *system of m linear equations in n unknowns* $x_1, x_2, \ldots, x_n$ is meant a system of the form

$$\text{(L)} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m \end{aligned}$$

where a_{ij} is a known real or complex number for each $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. By a *solution* of (L) is meant an n-tuple $(s_1, s_2, \ldots, s_n)$ of numbers $s_1, s_2, \ldots, s_n$ which as values of $x_1, x_2, \ldots, x_n$ respectively satisfy (L).

The subscripts on a_{ij} designate the location of that coefficient: i is the number of the equation and j is the number of the unknown where it appears. Hence, for example, a_{11} should be read "a, one, one" and a_{21} should be read "a, two, one"—not "a eleven" and "a twenty-one," respectively.

Except for problems, we shall tacitly restrict ourselves to systems of equations with real coefficients. Even though systems with real coefficients can have complex solutions we shall not utilize such solutions, so the term solution will refer to real solutions only. However, we make the observation that all the techniques and most of the remarks made about Gauss procedures in this chapter apply equally well to the complex case.

All systems of equations which have appeared thus far are linear. Examples of nonlinear systems are

$$\begin{aligned} x_1^2 + x_1x_2^3 - 2x_3 &= 1 \\ x_1x_3 + 2x_2^2 + x_3^4 &= 0 \end{aligned}$$

and

$$\sin x_1 + x_2 = 1$$

Next we wish to develop a more efficient notation for Gauss procedures. Consider the system

$$\text{(2.2)} \qquad \begin{aligned} 2x + y - z - w &= 6 \\ x + 2y - 2z + 4w &= 5 \\ x + y - z + 3w &= 4 \end{aligned}$$

The system

$$\text{(2.2}a\text{)} \qquad \begin{aligned} 2x_1 + x_2 - x_3 - x_4 &= 6 \\ x_1 + 2x_2 - 2x_3 + 4x_4 &= 5 \\ x_1 + x_2 - x_3 + 3x_4 &= 4 \end{aligned}$$

does not differ from (2.2) in any interesting way provided that (x_1, x_2, x_3, x_4)

is merely regarded as a notation for solutions of (2.2a) replacing the notation (x, y, z, w) for solutions of (2.2). That is, (2.2) and (2.2a) differ only in the names employed for the unknowns. This is an inessential difference, and to bypass it let us suppress unknowns entirely and write for (2.2) the array

$$\begin{bmatrix} 2 & 1 & -1 & -1 & \vert & 6 \\ 1 & 2 & -2 & 4 & \vert & 5 \\ 1 & 1 & -1 & 3 & \vert & 4 \end{bmatrix} \tag{2.3}$$

The array (2.3) is called a *matrix*. More completely, (2.3) is known as a 3×5 ("three by five") matrix since it has three *rows* and five *columns*. Obviously (2.3) presents at one stroke all the significant information applying to the systems (2.2) and (2.2a): the values of the coefficients and the values of the right sides. At the same time, the expression (2.3) is a more compact way of referring to these systems than either expression (2.2) or expression (2.2a).

(2.4) Definition. The **matrix of the system of linear equations**

$$\text{(L)} \qquad \begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = d_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = d_m \end{array}$$

is the $m \times (n + 1)$ array

$$\text{[L]} \qquad \begin{bmatrix} a_{11} & \cdots & a_{1n} & \vert & d_1 \\ \cdots & \cdots & \cdots & \vert & \cdot\cdot \\ a_{m1} & \cdots & a_{mn} & \vert & d_m \end{bmatrix}$$

which incorporates the coefficients and right-hand sides of the given system. [In some books [L] is called the "augmented" matrix of the system (L).]

Replacing a system of equations by its matrix consists basically in mere suppression of the names of the unknowns. Therefore it is not surprising that one can solve a system by simply applying a Gauss procedure directly to its matrix. Using $[k]$ to denote the kth row of a matrix as well as the kth equation of a system permits us to describe the basic operations (M), (A), and (G) practically as before:

- (M) Multiplying a row by a nonzero number: replacing $[k]$ by its c-multiple, $c[k]$
- (A) Adding a multiple of one row to another: replacing $[j]$ by $[j] + c[k]$
- (G) Replacing $[j]$ by the row $b[j] + c[k]$, where $b \neq 0$

(2.5) Example. We solve the system

$$\text{(L)} \qquad \begin{aligned} 2x_1 + x_2 - x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= -1 \\ 3x_1 \qquad - 3x_3 &= 2 \end{aligned}$$