INTRODUCTORY COLLEGE MATHEMATICS

HACKWORTH
and
HOWLAND

SAUNDERS SERIES IN MODULAR MATHEMATICS

Numeration

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PREFACE

Numeration

This book is one of the sixteen content modules in the Saunders Series in Modular Mathematics. The modules can be divided into three levels, the first of which requires only a working knowledge of arithmetic. The second level needs some elementary skills of algebra and the third level, knowledge comparable to the first two levels. *Numeration* is in level 2. The groupings according to difficulty are shown below.

Level 1	Level 2	Level 3
Tables and Graphs	*Numeration*	*Real Number System*
Consumer Mathematics	*Metric Measure*	*History of Real Numbers*
Algebra 1	*Probability*	*Indirect Measurement*
Sets and Logic	*Statistics*	*Algebra 2*
Geometry	*Geometric Measures*	*Computers*
		Linear Programming

The modules have been class tested in a variety of situations: large and small discussion groups, lecture classes, and in individualized study programs. The emphasis of all modules is upon ideas and concepts.

Numeration is appropriate for all non-science majors especially education and liberal arts students. It is also well suited for the mathematics majors and those students interested in computers.

Additive and multiplicative ancient and modern numeration systems are explained in the module before developing skills in converting numerals from one numeration system to another or from one base to another. The ability to add, subtract, multiply, and divide in bases other than base ten is promoted. Finally, the conversion of decimal and point-fractions to other bases is explained.

In preparing each module, we have been greatly aided by the valuable suggestions of the following excellent reviewers: William Andrews, Triton College, Ken Goldstein, Miami-Dade Community College, Don Hostetler, Mesa Community College, Karl Klee, Queensboro Community College, Pamela Matthews, Chabot College, Robert Nowlan, Southern Connecticut State College, Ken Seydel, Skyline College, Ara Sullenberger, Tarrant County Junior College, and Ruth Wing, Palm Beach Junior College. We thank them, and the staff at W. B. Saunders Company for their support.

Robert D. Hackworth
Joseph W. Howland

NOTE TO THE STUDENT

OBJECTIVES:

Upon completion of this module the reader is expected to be able to demonstrate the following skills and concepts:

1. To be able to find equivalent base ten numerals for numerals written in ancient or modern notation.

2. To be able to write numerals in ancient or modern base notation that are equivalent to base ten numerals.

3. To be able to add, subtract, multiply and divide in bases other than base ten.

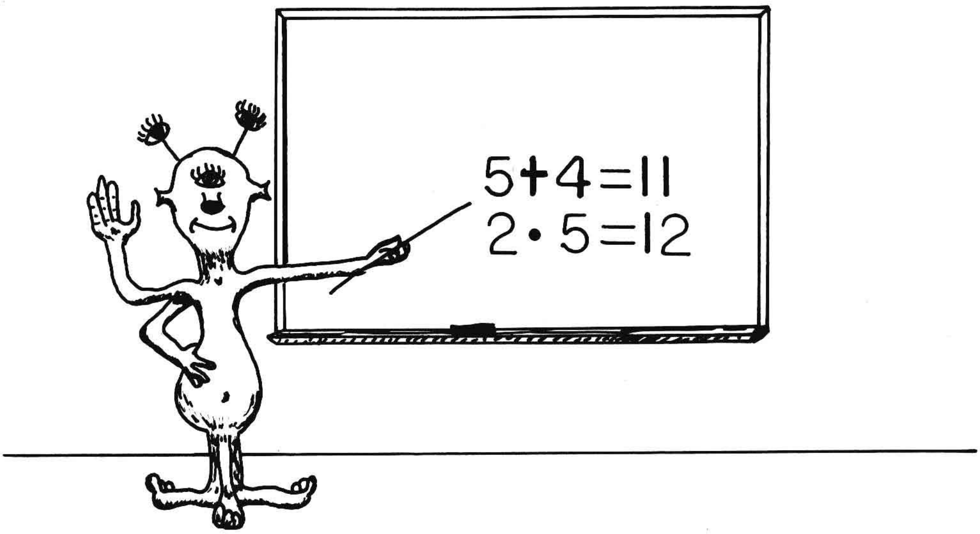
4. To be able to change decimal fractions to point fractions in other bases and vice versa.

Three types of problem sets with answers are included in this module. Progress Tests appear at the end of each section. These Progress Tests are always short. The questions asked in Progress Tests always come directly from the material of the section immediately preceding the test.

Exercise Sets appear less frequently in the module. More problems appear in an Exercise Set than in Progress Tests. The problems in Section I of the Exercise Sets are specifically chosen to match the objectives of the module. Problems in Section II are Challenge Problems.

A Self-Test is found at the end of the module. Self-Tests contain problems representative of the entire module.

In learning the material, the student is encouraged to try each problem set as it is encountered, checking answers and restudying those sections where difficulties are discovered. This procedure is guaranteed to be both efficient and effective.


5+4=11
2 • 5=12

CONTENTS

NUMERATION

INTRODUCTION

The space creature teaching arithmetic facts such as 5 + 4 = 11 and 5 • 2 = 12 is certainly different from most elementary school teachers. One of the major differences, which probably explains the arithmetic answers, is that the space creature has one thumb and three fingers on each hand. For such a creature a base eight system is likely to be developed; earth creatures probably developed base ten systems because of the number of thumbs and fingers of a human being.

In this module an explanation of other numeration systems is used to develop a better understanding of the familiar base ten, positional notation system.

The purpose of the module is to study methods by which non-negative rational numbers may be shown by the use of symbols. After the symbols used to designate numbers are studied, it will be possible to explain the close relationship between the symbols themselves and many problems in arithmetic.

Throughout this module, the only numbers that are being discussed are the non-negative rational numbers. The reader must avoid thinking or believing that any new or different numbers are being used. For example, suppose that "S" were a symbol for the number eight and "σ" were a symbol for the number ten. Then "S • σ" would be a name for the number eighty, because eight times ten is eighty regardless of any strange symbols that may be used to show the numbers. The concern here is only with the non-negative rational numbers; the names or symbols for these numbers may be changed but the numbers themselves remain the same.

The following distinction is made between number and numeral. Every non-negative rational number is an idea of quantity. An idea is an intangible. Nobody has ever seen an idea, spoken an idea, or written an idea. A sound, a mark, or a physical thing cannot itself be an idea, but only an effort to express the idea. Every non-negative rational number is an idea of quantity; no person has ever seen or written a non-negative rational number.

What do you see in this box? $\boxed{7}$ If you subscribe to the thoughts of the last paragraph, you must agree that you do not see the number 7 in the box. You see a particular mark on this paper. The symbol "7" is not itself an idea, but it is intended to express the idea of a number. The symbol "7" is called a numeral for the number seven.

The distinction being made between number and numeral is this: a number is an idea of quantity; it is an intangible. A numeral for a number is a symbol, word, or sound used to express the idea of a number; it is a physical thing.

It may seem that the distinction being made here between number and numeral is petty. In many contexts it could become so, but in this case the topic under consideration is numeration systems, and the intent is a study of how numerals are used to designate numbers.

A student in the United States, a farmer in China, a tailor in Germany, and even a Martian (if such exists) probably have about the same idea of the number seven. But the numerals used to express the number seven may be radically different. For example, 7, seven, 7, VII, 111, $\begin{matrix}1111\\111\end{matrix}$, $\begin{matrix}rrr\\rrrr\end{matrix}$, 21, siete, 5 + 2, and (13 - 6) are all symbols for the number 7 in some contexts. You would recognize some of these numerals as expressing the number seven, but others in the list may carry either no meaning or an entirely different meaning. The point to understand is that although the idea of the number seven may be universal, the numerals used to express this idea are in no way universal.

Progress Test 1

1. What is in the box? $\boxed{15}$ Is it a number or a numeral?

2. Explain the meaning of the following: Every non-negative real number is unique, but may have an infinite number of numerals.

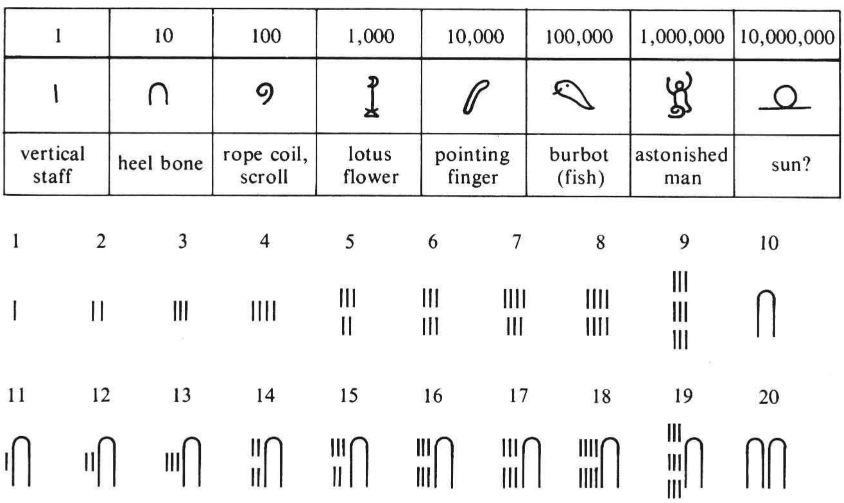
3. A space creature may be correct in writing 5 + 4 = 11 because 11 in a base eight system stands for nine. If the creature writes the true equality 5 • 2 = 12 the 12 in base eight stands for ________

4. 10 is a symbol for ten in a base ten system. 10 is a symbol for eight in a base eight system. 100 is a symbol for 10^2 in a base ten system. 100 is a symbol for ______ in a base eight system.

ADDITIVE NUMERATION SYSTEMS

Every numeral is dependent upon a particular culture, context, or use. A numeral is nothing more than an attempt to communicate the idea of number. As such, it is an agreement to accept some word or symbol as the expression of an idea rather than the idea itself. The numeral "3" does not look like the idea of three, but we have learned to look at the symbol and think the idea. The numeral "7" in no way accurately describes the idea of sevenness; in fact, the symbol "7" would be of no value if one were trying to express the idea of seven gallons to a person totally unfamiliar with our numeration system.

In the development of our numeral system, the descriptive qualities of the individual symbols have been lost. The numerals 4, 5, 6, 7, 8, and 9 do not look like the ideas of quantity they represent. The ancient Egyptian numeration system dating back to 3000 BC was a more descriptive system as shown by the following figure.



Hieroglyphic Symbols Used as Numerals by the Ancient Egyptians

The Egyptian system is called an additive numeration system because the number named by a numeral was the sum of the numbers named by the individual symbols. The system was also a base ten system because each power of ten (10, 100, 1000, 10000, etc.) up to 10,000,000 had a symbol.

The ancient Egyptian symbol for 2537 could be:

More than one correct Egyptian numeral for 2537 is a result of the fact that there was no prescribed order for listing the individual symbols. As long as there were two  's, five  's, three  's, and seven 's the numeral named 2537. This lack of ordering and the necessity of repeating many symbols made the Egyptian system awkward to work with.

Another additive system which is probably more familiar to the reader is the system of Roman numerals. The value of the individual symbols of the Romans is shown in the figure below.

1	5	10	50	100	500	1000
I	V	X	L	C	D	M

Like the Egyptians, the Romans had a base ten system with numerals for the powers of ten. Unlike the Egyptians, the Romans included other symbols such as V (five), L (fifty), and D (five hundred) to avoid the repetitions of five to nine symbols sometimes needed in the Egyptian system. Also, as an improvement upon the Egyptian system, the Romans imposed an ordering upon the individual symbols with the larger numerals to the left, arranged in descending order with the smaller numerals on the right. The Roman numeral for 2537 is shown below:

MMDXXXVII

The Roman system was an additive system, but it also included some uses of multiplication and subtraction. A numeral with a bar or vinculum placed above had its value multiplied by one thousand. For example, M is the symbol for 1000 and $\overline{M}$ is the symbol for a thousand thousands or 1,000,000.

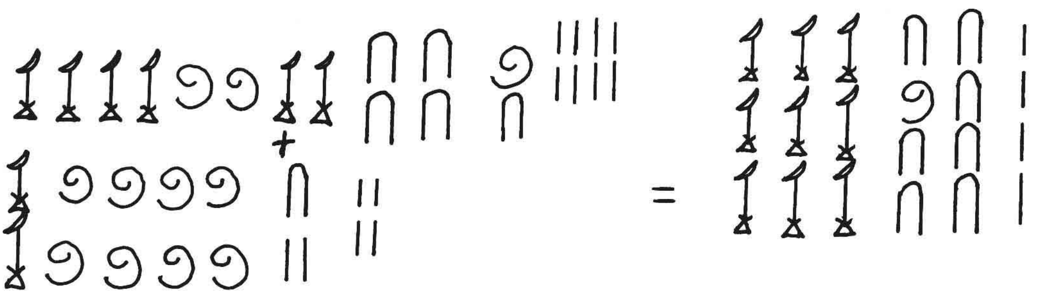
The use of V, L, and D eliminated some of the repetitions needed in the Egyptian system, but the Roman inclusion of subtraction was even more valuable in this regard. Whenever the ordering of larger to smaller (left to right) was violated by the Romans then the value of the smaller number symbol was to be subtracted from the value of the larger symbol on its right. For example, the numeral 994 could be written as:

CMXCIV

Reading from left to right: The first C precedes M and the combination CM means 1000 minus 100. The X precedes the C and the combination XC means 100 minus 10. The I precedes the V and the combination means 5 minus 1.

Although the numeration systems of the Egyptians and Romans were difficult to use with multiplication or division computations, the systems lend themselves easily to addition and subtraction.

Addition of Egyptian numerals simply requires putting together all the individual symbols. Wherever ten or more of a particular symbol results then each group of ten should be replaced by an equivalent symbol. The addition using Egyptian numerals for 6358 and 2816 is shown below.



The idea of "carrying" as it is taught in the modern elementary school is clearly seen in the example above. Eleven 100's (𓍢) are made into one 1000 (𓆼) and one 100 (𓍢). Similarly, fourteen 1's (𓏺) are made into one 10 (𓎆) and four 1's (𓏺).

Subtraction, in both the Egyptian and Roman systems, could be accomplished quite easily by taking one symbol from the minuend for each matching symbol in the subtrahend. The next example shows the process, including "borrowing," for 4369 minus 1857.

MMMMCCCLXIX - MDCCCLVII

4369 =	MMMM		CCC	L	X	
- 1857 =	M	D	CCC	L		VII
The problem is re-written or grouped as shown below.	MMM	DD	CCC	L	X	VIIII
	M	D	CCC	L		VII
	MM	D			X	II

MMMMCCCLXIX - MDCCCLVII = MMDXII

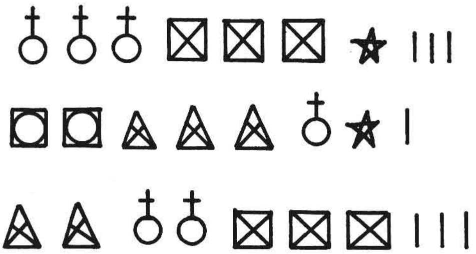
The borrowing consisted of making an M into 2 D's. The 9 (IX) was rewritten as VIIII for ease of understanding.

It is quite simple to create an additive system. Symbols are selected for the number 1, the number to be used as the base, and enough powers of the base to show all the numbers needed. For example, if a base five additive system were desired the following symbols may suffice.

1 5 25 125 625 3125

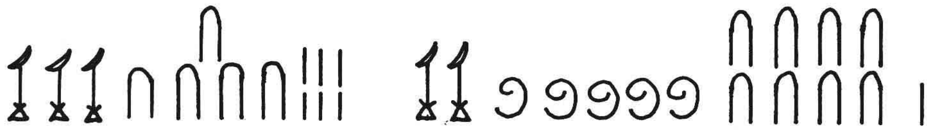
Using the Roman concept of ordering with the larger numerals to the left and smaller numerals to the right, additive numerals for 458, 8256, and 1578 would be written as shown below.

458 =
8256 =
1578 =



Progress Test 2

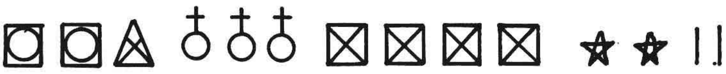
1. Add the following (3056 + 2581) using Egyptian numerals.



2. Subtract the following (5278 - 2647) using Roman numerals.

MMMMMCCLXXVIII - MMDCXLVII

3. What number (use a base ten numeral) is described by the following base five additive system numeral?



4. Find the numeral for 7439 in the base five additive system.

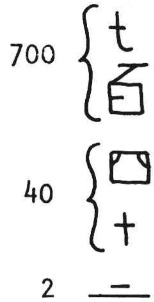
A MULTIPLICATIVE NUMERATION SYSTEM

The Egyptian and Roman numeration systems are called additive systems despite the fact that some multiplication and subtraction were involved in the Roman system. The primary characteristic of such additive systems is the repetition of symbols for quantities between two powers of the base. For example, thirty in the Roman system had a numeral XXX where X was repeated three times.

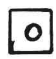
The Chinese and Japanese had a numeration system called a multiplicative system because it eliminated the need for repetition of symbols by installing multiplication as a basic part of the system. For example, the numeral for thirty in the Chinese-Japanese system consisted of the numeral for three written above the numeral for ten. The combination was read as three times ten or thirty.

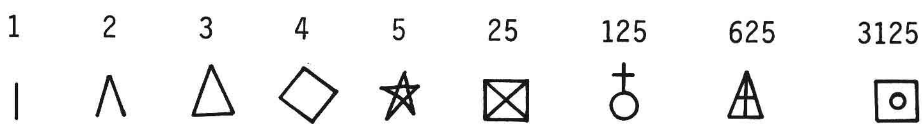
The symbols in the Chinese-Japanese system are shown below.

All numerals in the Chinese-Japanese system are written in vertical columns. The Chinese-Japanese numeral for 742 is shown to the right. Notice the manner in which multiplication is used in the example.

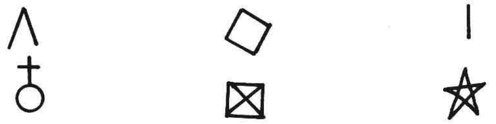


The basic set of symbols for a multiplicative system differs in one very important way from the symbols of an additive system. In an additive system the symbols must consist of a numeral for one and each desired power of the base. In a multiplicative system the symbols must also exist for each counting number less than the base. For example, the Chinese-Japanese system has symbols for 1, 2, 3, 4, 5, 6, 7, 8, 9, and all desired powers of the base, 10.

To make a multiplicative system using | for 1,  for 5,  for 25,  for 125,  for 625, and  for 3125 it is necessary to create symbols for 2, 3, and 4 which are the other counting numbers less than the base, 5. Suppose the following symbols are adopted for a multiplicative system.



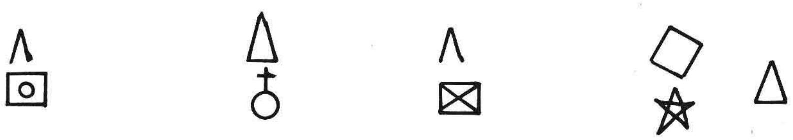
If each multiplication combination is written vertically, but all other symbols are arranged horizontally the numeral for 357 could be:



2 · 125 + 4 · 25 + 1 · 5 + 2

250 + 100 + 5 + 2

As another example, the multiplicative numeral for 6698 would be:



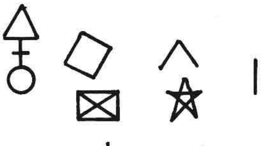
2 · 3125 + 3 · 125 + 2 · 25 + 4 · 5 + 3

6250 + 375 + 50 + 20 + 3

Notice that there is no use of the symbol  for 625 in the numeral above because no multiples are needed. There is no symbol for zero in this multiplicative system, but the need for zero is not evident in a multiplicative system because the power of the base was always shown with its multiplier. If a particular power of the base was not shown that was equivalent to having zero of that number.

Progress Test 3

1. Use the base five multiplicative system to find the number named by

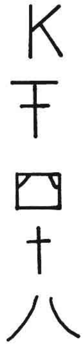
   a. 

   b. 

2. Write a base five multiplicative numeral for 89.

3. Write a base five multiplicative numeral for 268.

4. Use the Chinese-Japanese numeration system to find the number for the figure on the right.



5. Write 7,926 in Chinese-Japanese numerals.

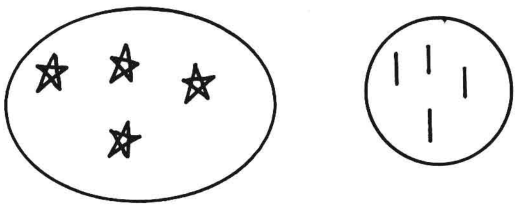
Exercise Set 1

I. 1. If  = 9, T = 8, S = 9 and  = 8, is  · T =  · S?

2. True or false? Many numerals can represent the same number.

3. Is the contents of  a numeral or a number?

4. The groupings below have the common concept of 4'ness. Is the common concept a number or a numeral?



5. 3 + 4 = 11 is correct in base 6, because 11 in base 6 represents 7 in base 10. 2 · 4 = 12 in base 6 because 12 represents ________ in the base 10.

6. True or false: Numbers can be erased.

7. In base 5, 3 + 3 = 11 is correct because 11 stands for 6 in base 5. 4 · 2 = 13 is correct in base 5 because 13 base five stands for ________ in base 10.