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SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

COLLEGE MATHEMATICS
Second Edition

Algebra

Discrete Mathematics

Trigonometry

Geometry

Introduction to Calculus

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McGRAW-HILL, INC.

New York St. Louis San Francisco Auckland Bogotá Caracas Lisbon London Madrid Mexico City Milan Montreal New Delhi San Juan Singapore Sydney Tokyo Toronto

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Schaum's Outline of Theory and Problems of
COLLEGE MATHEMATICS

3 4 5 6 7 8 9 10 11 12 13 14 15 SH SH 9 8 7 6 5 4

ISBN 0-07-002664-5

Sponsoring Editor: John Aliano
Production Supervisor: Leroy Young
Editing Supervisors: Meg Tobin, Maureen Walker
Cover design by Amy E. Becker.

This book is printed on recycled paper containing a minimum of 50% total recycled fiber with 10% postconsumer de-inked fiber. Soybean based inks are used on the cover and text.

Library of Congress Cataloging-in-Publication Data

Ayres, Frank, date
Schaum's outline of theory and problems of college mathematics.—2nd ed. / Frank Ayres, Jr., Philip A. Schmidt.
p. cm.—(Schaum's outline series)
Rev. ed. of: First year college mathematics / Frank Ayres, Jr. c 1958.
Includes index.
ISBN 0-07-002664-5
1. Mathematics. 2. Mathematics—Problems, exercises, etc. I. Schmidt, Philip A. II. Ayres, Frank, date First year college mathematics. III. Title. IV. Title: Theory and problems of college mathematics. V. Title: College mathematics.
QA37.2.A948 1992
510—dc20 91-19897
CIP

Preface

As Dr. Ayres indicates in his preface to the First Edition, this book will help students become proficient in the mathematics commonly presented in the first year (or two) of college. In updating this text, I have left that point-of-view intact; however, I have deleted material that is no longer presented (for example, logarithmic solutions of the right triangle). Additionally, I have added material which has been placed in the curriculum since the publication of the first edition and I have "modernized" many of the problems and exercises. The notation has been changed when necessary and discrete mathematics has been reemphasized.

My thanks must be expressed to Professor Ayres: He has provided me (and so many students) with the very finest of review materials. My thanks, as always, go to John Aliano, Executive Editor of the Schaum Division at McGraw-Hill and to Maureen Walker for her handling of the manuscript and proofs. Cathy Decker-Coffey typed all revisions with her usual meticulous care. Finally, my family has provided me with "quiet time" at home and without that contribution, this revision would have been impossible.

PHILIP A. SCHMIDT
New Paltz, NY
January 1992

Preface to the First Edition

This book is designed primarily to assist students in acquiring a more thorough knowledge and proficiency in basic college mathematics. It includes a thorough coverage of algebra, plane trigonometry, and plane analytic geometry together with selected topics in solid analytic geometry and a brief introduction to the calculus, in that order. In addition to the use of the book by students taking a formal course in first year college mathematics, it should also be of considerable value to those who wish to review the fundamental principles and applications in anticipation of further work in mathematics.

Each chapter begins with a clear statement of the pertinent definitions, principles, and theorems, together with illustrative and descriptive material. This is followed by carefully graded sets of solved and supplementary problems. The solved problems have been selected and solutions arranged so that a study of each will be rewarding. They serve to illustrate and amplify the theory, provide the repetition of basic principles so vital to effective teaching, and bring into sharp focus those fine points without which the student continually feels on unsafe ground. Derivations of formulas and proofs of theorems are included among the solved problems. The supplementary problems offer a complete review of the material of each chapter.

Although in many texts some degree of unification of the material has been achieved, it seemed best to make no attempt in that direction here. However, the reader will find that the material has been so divided into chapters and the problems in these chapters so arranged as to make the book a useful supplement to all current standard texts.

Considerably more material has been included here than can be covered in most first courses. This has been done to make the book more flexible, to provide a more useful book of reference, and to stimulate further interest in the topics.

The author gratefully acknowledges his indebtedness to Mr. Henry Hayden for painstaking work in the preparation of all drawings and for typographical arrangement.

FRANK AYRES, JR.
Carlisle, Pa.
June 1958

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Chapter 1

The Number System of Algebra

ELEMENTARY MATHEMATICS is concerned mainly with certain elements called *numbers* and with certain operations defined on them.

The unending set of symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ... used in counting are called *natural numbers.*

In adding two of these numbers, say 5 and 7, we begin with 5 (or with 7) and count to the *right* seven (or five) numbers to get 12. The sum of two natural numbers is a natural number, that is, the sum of two members of the above set is a member of the set.

In subtracting 5 from 7, we begin with 7 and count to the *left* five numbers to 2. It is clear, however, that 7 cannot be subtracted from 5 since there are only four numbers to the left of 5.

INTEGERS. In order that subtraction be always possible, it is necessary to increase our set of numbers. We prefix each natural number with a + sign (in practice, it is more convenient not to write the sign) to form the *positive integers,* we prefix each natural number with a − sign (the sign must always be written) to form the *negative integers,* and we create a new symbol 0, read *zero.* On the set of *integers*

$$\ldots, -8, -7, -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6, +7, +8, \ldots$$

the operations of addition and subtraction are possible without exception.

To add two integers as +7 and −5, we begin with +7 and count to the left (indicated by the sign of −5) five numbers to +2 or we begin with −5 and count to the right (indicated by the sign of +7) seven numbers to +2. How would you add −7 and −5?

To subtract +7 from −5 we begin with −5 and count to the left (opposite to the direction indicated by +7) seven numbers to −12. To subtract −5 from +7 we begin with +7 and count to the right (opposite to the direction indicated by −5) five numbers to +12. How would you subtract +7 from +5? −7 from −5? −5 from −7?

If one is to operate easily with integers it is necessary to avoid the process of counting. To do this we memorize an addition table and establish certain rules of procedure. We note that each of the numbers +7 and −7 is seven steps from 0 and indicate this fact by saying that the *numerical value* of each of the numbers +7 and −7 is 7. We may state:

Rule 1. To add two numbers having like signs, add their numerical values and prefix their common sign.

Rule 2. To add two numbers having unlike signs, subtract the smaller numerical value from the larger, and prefix the sign of the number having the larger numerical value.

Rule 3. To subtract a number, change its sign and add. Since $3 \cdot 2 = 2 + 2 + 2 = 3 + 3 = 6$, we assume

$$(+3)(+2) = +6 \qquad (-3)(+2) = (+3)(-2) = -6 \qquad \text{and} \qquad (-3)(-2) = +6$$

Rule 4. To multiply or divide two numbers (never divide by 0!), multiply or divide the numerical values, prefixing a + sign if the two numbers have like signs and a − sign if the two numbers have unlike signs. (See Problem 1.1.)

If m and n are integers then $m + n$, $m - n$, and $m \cdot n$ are integers but $m \div n$ may not be an integer. (Common fractions will be treated in the next section.) Moreover, there exists a unique integer x such that $m + x = n$. If $x = 0$, then $m = n$; if x is positive ($x > 0$), then m is less than n ($m < n$); if x is negative ($x < 0$), then m is greater than n ($m > n$).

The integers may be made to correspond one-to-one with equally spaced points on a straight line as in Fig. 1-1. Then $m > n$ indicates that the point on the scale corresponding to m lies to the right of the point corresponding to n. There will be no possibility of confusion if we write the point m rather than the point which corresponds to m and we shall do so hereafter. Then $m < n$ indicates that the point m lies to the left of n. (See Problems 1.2–1.4.)

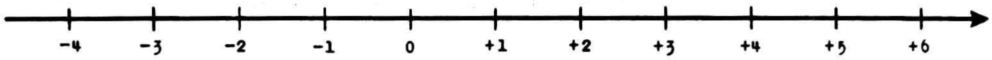

Fig. 1-1

Every positive integer m is divisible by ± 1 and $\pm m$. A positive integer $m > 1$ is called a prime if its only factors or divisors are ± 1 and $\pm m$; otherwise, m is called *composite*. For example, 2, 7, 19, are primes while $6 = 2 \cdot 3$, $18 = 2 \cdot 3 \cdot 3$, and $30 = 2 \cdot 3 \cdot 5$ are composites. In these examples, the composite numbers have been expressed as products of prime factors, that is, factors which are prime numbers. Clearly, if $m = r \cdot s \cdot t$ is such a factorization of m, then $-m = (-1)r \cdot s \cdot t$ is a factorization of $-m$. (See Problems 1.5–1.6.)

THE RATIONAL NUMBERS. The set of rational numbers consists of all numbers of the form m/n, where m and $n \neq 0$ are integers. Thus, the rational numbers include the integers and common fractions.

Every rational number has an infinitude of representations; for example, the integer 1 may be represented by $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots$ and the fraction $\frac{2}{3}$ may be represented by $\frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots$ A fraction is said to be expressed in lowest terms by the representation m/n when m and n have no common prime factor. The most useful rule concerning rational numbers is, therefore,

Rule 5. The value of a rational number is unchanged if both the numerator and denominator are multiplied or divided by the same nonzero number.

Caution. We use Rule 5 with division to reduce a fraction to lowest terms. For example, we write $\frac{15}{21} = \frac{\cancel{3} \cdot 5}{\cancel{3} \cdot 7} = \frac{5}{7}$ and speak of canceling the 3's. Now canceling is not an operation on numbers. We cancel or strike out the 3's as a safety measure, that is, to be sure that they will not be used in computing the final result. The operation is division and Rule 5 states that we may divide the numerator by 3 provided we also divide the denominator by 3. This point is belabored here because of the all too common error $\frac{12\cancel{a} - 5}{7\cancel{a}}$. The fact is that $\frac{12a - 5}{7a}$ cannot be further simplified for if we divide $7a$ by a we must also divide $12a$ and 5 by a. This would lead to the more cumbersome $\frac{12 - 5/a}{7}$. (See Problems 1.7–1.8.)

The rational numbers may be associated in a one-to-one manner with points on a straight line as in Fig. 1-2. Here the point associated with the rational number m is m units from that point (called the origin) associated with 0, the distance between the points 0 and 1 being the unit of measure.

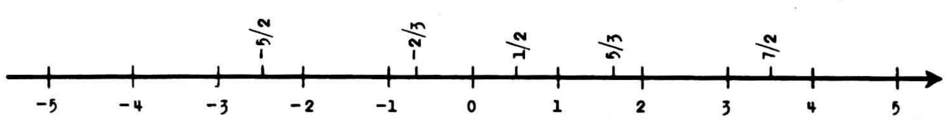

Fig. 1-2

If two rational numbers have representations r/n and s/n, where n is a positive integer, then $r/n > s/n$ if $r > s$, $r/n = s/n$ if $r = s$, and $r/n < s/n$ if $r < s$. Thus, in comparing two rational numbers it is necessary to express them with the same denominator. Of the many denominators (positive integers) there is always a least one, called the *least common denominator.* For the fractions $\frac{3}{5}$ and $\frac{2}{3}$, the least common denominator is 15. We conclude that $\frac{3}{5} < \frac{2}{3}$ since $\frac{3}{5} = \frac{9}{15} < \frac{10}{15} = \frac{2}{3}$. (See Problems 1.9–1.10.)

Rule 6. The sum (difference) of two rational numbers expressed with the same denominator is a rational number whose denominator is the common denominator and whose numerator is the sum (difference) of the numerators.

Rule 7. The product of two or more rational numbers is a rational number whose numerator is the product of the numerators and whose denominator is the product of the denominators of the several factors.

Rule 8. The quotient of two rational numbers can be evaluated by the use of Rule 5 with the least common denominator of the two numbers as the multiplier.

(See Problems 1.11–1.13.)

If a and b are rational numbers, $a + b$, $a - b$, and $a \cdot b$ are rational numbers. Moreover, if a and b are $\neq 0$, there exists a rational number x, unique except for its representation, such that

$$ax = b \tag{1.1}$$

When a or b or both are zero, we have the following situations:

$b = 0$ and $a \neq 0$: (*1.1*) becomes $a \cdot x = 0$ and $x = 0$, that is, $0/a = 0$ when $a \neq 0$.

$a = 0$ and $b \neq 0$: (*1.1*) becomes $0 \cdot x = b$; then $b/0$, when $b \neq 0$, is without meaning since $0 \cdot x = 0$.

$a = 0$ and $b = 0$: (*1.1*) becomes $0 \cdot x = 0$; then $0/0$ is indeterminate since every number x satisfies the equation.

In brief: $0/a = 0$ when $a \neq 0$, but division by 0 is never permitted.

DECIMALS. In writing numbers we use a positional system, that is, the value given any particular digit depends upon its position in the sequence. For example, in 423 the positional value of the digit 4 is 4(100) while in 234 the positional value of the digit 4 is 4(1). Since the positional value of a digit involves the number 10, this system of notation is called the *decimal system.* In this system the number 4238.75 means

$$4(1000) + 2(100) + 3(10) + 8(1) + 7(\tfrac{1}{10}) + 5(\tfrac{1}{100})$$

It is interesting to note that from this example certain definitions to be made in a later study of exponents may be anticipated. Since $1000 = 10^3$, $100 = 10^2$, $10 = 10^1$ it would seem natural to define $1 = 10^0$, $\frac{1}{10} = 10^{-1}$, $\frac{1}{100} = 10^{-2}$.

By the process of division, any rational number can be expressed as a decimal; for example, $\frac{70}{33} = 2.121212\ldots$. This is termed a *repeating decimal* since the digits 12, called the cycle, are repeated without end. It will be seen later that every repeating decimal represents a rational number.

In operating with decimals, it is necessary to "round off" a decimal representation to a prescribed number of decimal places. For example, $\frac{1}{3} = 0.3333\ldots$ is written as 0.33 to two decimal places and $\frac{2}{3} = 0.6666\ldots$ is written as 0.667 to three decimal places. In rounding off, use will be made of the Computer's Rule:

(*a*) Increase the last digit retained by 1 if the digits rejected exceed the sequence 50000 . . . ; for example, 2.384629 . . . becomes 2.385 to three decimal places.

(*b*) Leave the last digit retained unchanged if the digits rejected are less than 5000 . . . ; for example, 2.384629 . . . becomes 2.38 to two decimal places.

(*c*) Make the last digit retained even if the digit rejected is exactly 5; for example, to three decimal places 11.3865 becomes 11.386 and 9.3815 becomes 9.382.

(See Problem 1.14.)

PERCENTAGE. The symbol %, read percent, means per hundred; thus 5% is equivalent to $\frac{5}{100}$ or 0.05.

Any number, when expressed in decimal notation, can be written as a percent by multiplying by 100 and adding the symbol %. For example, $0.0125 = 100(0.0125)\% = 1.25\% = 1\frac{1}{4}\%$, $2.3 = 230\%$, and $\frac{7}{20} = 0.35 = 35\%$.

Conversely, any percentage may be expressed in decimal form by dropping the symbol % and dividing by 100. For example, $42.5\% = 42.5/100 = 0.425$, $3.25\% = 0.0325$, and $2000\% = 20$.

When reckoning percentages, express the percent as a decimal and, when possible, as a simple fraction. For example, $4\frac{1}{4}\%$ of $48 = 0.0425 \times 48 = 2.04$ and $12\frac{1}{2}\%$ of $5.28 = \frac{1}{8}$ of $5.28 = 0.66$. (See Problems 1.15–1.18.)

THE IRRATIONAL NUMBERS. The existence of numbers other than the rational numbers may be inferred from either of the following considerations:

(*a*) We may conceive of a nonrepeating decimal constructed in endless time by setting down a succession of digits chosen at random.

(*b*) The length of the diagonal of a square of side 1 is not a rational number, that is, there exists no rational number a such that $a^2 = 2$. Numbers such as $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[5]{-3}$, and π (but not $\sqrt{-3}$ or $\sqrt[4]{-5}$) are called *irrational numbers.* The first three of these are called *radicals.* The radical $\sqrt[n]{a}$ is said to be of order n; n is called the *index* and a is called the *radicand.*

(See Problems 1.19–1.21.)

THE REAL NUMBERS. The set of *real numbers* consists of the rational and irrational numbers. The real numbers may be ordered by comparing their decimal representations. For example, $\sqrt{2} = 1.4142\ldots$; then $\frac{7}{5} = 1.4 < \sqrt{2}$, $\frac{3}{2} = 1.5 > \sqrt{2}$, etc.

We assume that the totality of real numbers may be placed in one-to-one correspondence with the totality of points on a straight line. See Fig. 1-3.

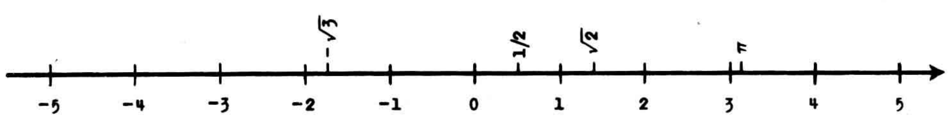

Fig. 1-3

The number associated with a point on the line, called the *coordinate* of the point, gives its distance and direction from that point (called the origin) associated with the number 0. If a point A has coordinate a, we shall speak of it as the point $A\,(a)$.

The directed distance from point $A\ (a)$ to point $B\ (b)$ on the real number scale is given by $AB = b - a$. The midpoint of the segment AB has coordinate $\frac{1}{2}(a + b)$. (See Problems 1.22–1.25.)

THE COMPLEX NUMBERS. In the set of real numbers there is no number whose square is -1. If there is to be such a number, say $\sqrt{-1}$ then by definition $(\sqrt{-1})^2 = -1$. Note carefully that $(\sqrt{-1})^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$ is incorrect. In order to avoid this error, the symbol i with the following properties is used:

$$\text{If } a > 0, \quad \sqrt{-a} = i\sqrt{a}; \qquad i^2 = -1$$

Then $$(\sqrt{-2})^2 = \sqrt{-2}\,\sqrt{-2} = (i\sqrt{2})(i\sqrt{2}) = i^2 \cdot 2 = -2$$

and $$\sqrt{-2}\,\sqrt{-3} = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}$$

Numbers of the form $a + bi$, where a and b are real numbers, are called *complex numbers*. In the complex number $a + bi$, a is called the *real part* and bi is called the *imaginary part*. Numbers of the form ci, where c is real, are called *imaginary numbers*.

The complex number $a + bi$ is a real number when $b = 0$ and a pure imaginary number when $a = 0$. When a complex number is not a real number it is called *imaginary*.

Complex numbers will be considered in more detail in a later chapter. Only the following operations will be considered here:

To add (subtract) two complex numbers, add (subtract) the real parts and add (subtract) the pure imaginary parts.

To multiply two complex numbers, form the product treating i as an ordinary number and then replace i^2 by -1.

(See Problems 1.26–1.27.)

Solved Problems

1.1 Give the results when the following operations are performed on each of the numbers $-9, -6, -3, 0, 3, 6, 9, 12, 15$: (*a*) add -4, (*b*) subtract 6, (*c*) subtract -2, (*d*) multiply by -5, (*e*) divide by 3, (*f*) divide by -1, (*g*) divide by -3.

(*a*) $-13, -10, -7, -4, -1, 2, 5, 8, 11$

(*b*) $-15, -12, -9, -6, -3, 0, 3, 6, 9$

(*c*) $-7, -4, -1, 2, 5, 8, 11, 14, 17$

(*d*) $45, 30, 15, 0, -15, -30, -45, -60, -75$

(*e*) $-3, -2, -1, 0, 1, 2, 3, 4, 5$

(*f*) $9, 6, 3, 0, -3, -6, -9, -12, -15$

(*g*) $3, 2, 1, 0, -1, -2, -3, -4, -5$

1.2 Arrange the integers in each set so that they may be separated by $<$ and again so that they may be separated by $>$.

(*a*) $3, 15, 12, 20, 0$ *Ans.* $0 < 3 < 12 < 15 < 20$; $20 > 15 > 12 > 3 > 0$

(*b*) $3, -3, 5, 0, -2$ *Ans.* $-3 < -2 < 0 < 3 < 5$; $5 > 3 > 0 > -2 > -3$

(*c*) $-7, -5, -10, -8$ *Ans.* $-10 < -8 < -7 < -5$; $-5 > -7 > -8 > -10$

1.3 Let x be an integer. By means of Fig. 1-1, interpret each of the following:

(*a*) $x < 10$ *Ans.* x is to the left of 10.

(*b*) $x > -2$ *Ans.* x is to the right of -2.

(*c*) $x \geq 5$ *Ans.* x is 5 or is to the right of 5.

(*d*) $2 < x < 6$ *Ans.* x is to the right of 2 but to the left of 6.

(*e*) $10 > x > -3$ *Ans.* x is to the left of 10 but to the right of -3.

1.4 List all integral values of x when

(*a*) $2 < x < 6$ *Ans.* 3, 4, 5

(*b*) $2 > x > -3$ *Ans.* $-2, -1, 0, 1$

(*c*) $-5 < x < 0$ *Ans.* $-4, -3, -2, -1$

(*d*) $2 \leq x < 5$ *Ans.* 2, 3, 4

(*e*) $-4 < x \leq -1$ *Ans.* $-3, -2, -1$

(*f*) $2 \geq x \geq -3$ *Ans.* $-3, -2, -1, 0, 1, 2$

1.5 List the first 15 primes.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

1.6 Express each of the following integers as a product of primes: (*a*) 6930, (*b*) 23,595.

(*a*) A systematic procedure is to test the primes 2, 3, 5, ... in order. When a factor is found, we then repeat the procedure, using in order all primes not already rejected, on the quotient. Thus, $6930 = 2.3465$. Since 3465 is not divisible by 2, we try 3 and obtain $6930 = 2 \cdot 3 \cdot 1155$. Using 3 again, we find $6930 = 2 \cdot 3 \cdot 3 \cdot 385$. Since 385 is not divisible by 3, we try 5 and obtain $6930 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 77 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$.

(*b*) $23{,}595 = 3 \cdot 5 \cdot 11 \cdot 11 \cdot 13$.

1.7 Express $\frac{5}{6}$ as a fraction having denominator (*a*) 12, (*b*) 36, (*c*) 84, (*d*) 126.

(*a*) $\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}$ (*b*) $\frac{5}{6} = \frac{5 \cdot 6}{6 \cdot 6} = \frac{30}{36}$ (*c*) $\frac{5}{6} = \frac{5 \cdot 14}{6 \cdot 14} = \frac{70}{84}$ (*d*) $\frac{5}{6} = \frac{5 \cdot 21}{6 \cdot 21} = \frac{105}{126}$

1.8 Reduce to lowest terms: (*a*) $\frac{6}{24}$, (*b*) $\frac{30}{42}$, (*c*) $\frac{27}{45}$, (*d*) $\frac{60}{96}$.

(*a*) $\frac{6}{24} = \frac{2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3} = \frac{1}{2 \cdot 2} = \frac{1}{4}$ (*c*) $\frac{27}{45} = \frac{3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 5} = \frac{3}{5}$

(*b*) $\frac{30}{42} = \frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 7} = \frac{5}{7}$ (*d*) $\frac{60}{96} = \frac{2 \cdot 2 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = \frac{5}{2 \cdot 2 \cdot 2} = \frac{5}{8}$

1.9 In each of the following find the lowest common denominator (LCD) of the several fractions: (*a*) $\frac{3}{4}, \frac{5}{6}$, (*b*) $\frac{1}{6}, \frac{2}{9}, \frac{5}{24}$, (*c*) $\frac{1}{12}, \frac{7}{60}, \frac{2}{25}$, (*d*) $\frac{7}{72}, \frac{4}{75}, \frac{9}{80}$.

To find the LCD: Express each of the several denominators as the product of prime factors, write each distinct factor the greatest number of times it occurs in any denominator, and form the product.

(*a*) Here $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$; LCD $= (2 \cdot 2)(3) = 12$.

(*b*) Here $6 = 2 \cdot 3$, $9 = 3 \cdot 3$, $24 = 2 \cdot 2 \cdot 2 \cdot 3$; LCD $= (2 \cdot 2 \cdot 2)(3 \cdot 3) = 72$.

(*c*) Here $12 = 2 \cdot 2 \cdot 3$, $60 = 2 \cdot 2 \cdot 3 \cdot 5$, $25 = 5 \cdot 5$; LCD $= (2 \cdot 2)(3)(5 \cdot 5) = 300$.

(*d*) Here $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$, $75 = 3 \cdot 5 \cdot 5$, $80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$; LCD $= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 = 3600$.

1.10 Arrange each set of rational numbers so that they may be separated by $<$: (*a*) $1, -\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, -\frac{5}{3}$; (*b*) $-\frac{2}{3}, -\frac{7}{8}, -\frac{5}{6}, -\frac{11}{6}, -\frac{5}{12}$.

(*a*) Since $1 - \frac{12}{12}$, $-\frac{1}{2} = -\frac{6}{12}$, $\frac{1}{3} = \frac{4}{12}$, $\frac{3}{4} = \frac{9}{12}$, $-\frac{5}{3} = -\frac{20}{12}$, then $-\frac{5}{3} < -\frac{1}{2} < \frac{1}{3} < \frac{3}{4} < 1$.

(*b*) Since $-\frac{2}{3} = -\frac{16}{24}$, $-\frac{7}{8} = -\frac{21}{24}$, $-\frac{5}{6} = -\frac{20}{24}$, $-\frac{11}{6} = -\frac{44}{24}$, $-\frac{5}{12} = -\frac{10}{24}$, then $-\frac{11}{6} < -\frac{7}{8} < -\frac{5}{6} < -\frac{2}{3} < -\frac{5}{12}$.

1.11 Perform the indicated operations:

(*a*) $\frac{1}{2} + \frac{2}{5} = \frac{5}{10} + \frac{4}{10} = \frac{5+4}{10} = \frac{9}{10}$

(*b*) $\frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$

(*c*) $\frac{11}{12} + \frac{4}{3} - \frac{1}{6} = \frac{11+16-2}{12} = \frac{25}{12}$

(*d*) $\frac{15}{64} - \frac{17}{32} + \frac{1}{8} = \frac{15-34+8}{64} = -\frac{11}{64}$

(*e*) $\frac{2}{3} \times \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$

(*f*) $\frac{5}{8} \times \frac{16}{15} = \frac{5 \cdot 16}{8 \cdot 15} = \frac{2}{3}$

(*g*) $\frac{15}{7}\left(-\frac{21}{10}\right)\left(\frac{4}{9}\right) = -\frac{15 \cdot 21 \cdot 4}{7 \cdot 10 \cdot 9} = -\frac{3 \cdot 5 \cdot 3 \cdot 7 \cdot 2 \cdot 2}{7 \cdot 2 \cdot 5 \cdot 3 \cdot 3} = -2$

(*h*) $\frac{5}{7} \div \frac{11}{14} = 14\frac{5}{7} \div 14\frac{11}{14} = 10 \div 11 = \frac{10}{11}$

(*i*) $\frac{16}{5} \div \frac{7}{10} = 10\frac{16}{5} \div 10\frac{7}{10} = \frac{32}{7}$

(*j*) $8 \div \left(-\frac{2}{3}\right) = 3 \cdot 8 \div 3\left(-\frac{2}{3}\right) = -\frac{24}{2} = -12$

(*k*) $\frac{4 - \frac{3}{4}}{2 + \frac{1}{2}} = \frac{4 \cdot 4 - 4(\frac{3}{4})}{4 \cdot 2 + 4(\frac{1}{2})} = \frac{16-3}{8+2} = \frac{13}{10}$

(*l*) $\frac{\frac{1}{2} - \frac{7}{4}}{1 - \frac{3}{8}} = \frac{8(\frac{1}{2}) - 8(\frac{7}{4})}{8 \cdot 1 - 8(\frac{3}{8})} = \frac{4-14}{8-3} = \frac{-10}{5} = -2$

1.12 The product of five factors each equal to 2, that is, $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, is denoted by 2^5 and read the fifth power of 2. We call 2 the *base* and 5 the *exponent.* Show that the solutions of Solved Problem 8(*a*), (*c*), (*d*) may be written as follows using exponents:

$$(a)\ \frac{6}{24} = \frac{2 \cdot 3}{2^3 \cdot 3} = \frac{1}{2^2} = \frac{1}{4} \qquad (c)\ \frac{27}{45} = \frac{3^3}{3^2 \cdot 5} = \frac{3}{5} \qquad (d)\ \frac{60}{96} = \frac{2^2 \cdot 3 \cdot 5}{2^5 \cdot 3} = \frac{5}{2^3} = \frac{5}{8}$$

1.13 Verify

$$(a)\ a^4 \cdot a^2 = (a \cdot a \cdot a \cdot a)(a \cdot a) = a^{4+2} = a^6, \qquad (b)\ \frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a^{5-3} = a^2,$$

$$(c)\ \frac{a^3}{a^5} = \frac{a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a^{5-3}} = \frac{1}{a^2},$$

$$(d)\ (a \cdot b)^4 = (a \cdot b)(a \cdot b)(a \cdot b)(a \cdot b) = (a \cdot a \cdot a \cdot a)(b \cdot b \cdot b \cdot b) = a^4b^4.$$

The general rules are: If m and n are positive integers, then $a^m a^n = a^{m+n}$; $a^m/a^n = a^{m-n}$ if $m > n$; $a^m/a^n = 1$ if $m = n$; $a^m/a^n = 1/a^{n-m}$ if $m < n$; $(a \cdot b)^m = a^m b^m$.

1.14 Express $\frac{2}{7}$ as a decimal to (*a*) five, (*b*) four, (*c*) three, (*d*) two decimal places.

By division, $\frac{2}{7} = 0.285714\ldots$. Then we have for *a*) 0.28571, *b*) 0.2857, *c*) 0.286, *d*) 0.29.

1.15 Compute:

(*a*) 6% of 400 = $0.06 \times 400 = 24$ (*c*) 135% of 500 = $1.35 \times 500 = 675$

(*b*) $4\frac{1}{2}$% of 1200 = $0.045 \times 1200 = 54$ (*d*) 2% of 6% of 8000 = $0.02 \times 0.06 \times 8000 = 9.6$

1.16 What percent (*a*) of 75 is 15? (*b*) of 112 is 14? (*c*) of 72 is 3.96? (*d*) of 0.44 is 1.034?

(*a*) $\frac{15}{75} = \frac{1}{5} = 20\%$ (*c*) $3.96/72 = 0.055 = 5\frac{1}{2}\%$

(*b*) $\frac{14}{112} = \frac{1}{8} = 12\frac{1}{2}\%$ (*d*) $1.034/0.44 = 2.35 = 235\%$

1.17 Find the number, given (*a*) 5% of it is 32, (*b*) 8% of it is 8.4, (*c*) 210% of it is 54.6, (*d*) 0.5% of it is 2.3.

(*a*) 1% of the number is $\frac{32}{5} = 6.4$; 100% of the number is $100 \times 6.4 = 640$ or $32/0.05 = 640$.

(*b*) $8.4/0.08 = 105$ (*c*) $54.6/2.1 = 26$ (*d*) $2.3/0.005 = 460$

1.18 Express the *percentage* strength of each of the following solutions (by a 10% silver nitrate solution is meant 10 grams of silver nitrate in 100 grams of solution): (*a*) 200 grams of solution containing a 0.5-gram tablet of bichloride of mercury; (*b*) 50 grams of solution containing 0.8 gram of salt.

(*a*) $0.5/200 = 0.0025 = 0.25\% = \frac{1}{4}\%$ (*b*) $0.8/50 = 0.016 = 1.6\%$

1.19 Simplify each of the following radicals:

$$(a)\ \sqrt{700} = \sqrt{100 \cdot 7} = 10\sqrt{7} \qquad (e)\ \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{2}{3}\sqrt{3}$$

$$(b)\ \sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = 2\sqrt[3]{3} \qquad (f)\ \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\sqrt{2 \cdot 3}}{3} = \frac{\sqrt{6}}{3}$$

$$(c)\ \sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2} \qquad (g)\ \sqrt{\frac{5}{8}} = \sqrt{\frac{5 \cdot 2}{8 \cdot 2}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{4}$$

$$(d)\ \sqrt[5]{-64} = \sqrt[5]{-32 \cdot 2} = -2\sqrt[5]{2}$$

1.20 Perform the indicated operations.

(*a*) $4\sqrt{2}+3\sqrt{2}-2\sqrt{2}=(4+3-2)\sqrt{2}= \ =5\sqrt{2}$

(*b*) $6\sqrt{3}-\sqrt{27}=6\sqrt{3}-3\sqrt{3}=3\sqrt{3}$

(*c*) $2\sqrt[3]{5}-\sqrt[3]{135}+4\sqrt[6]{25}=2\sqrt[3]{5}-3\sqrt[3]{5}+4\sqrt[3]{5}=3\sqrt[3]{5}$

(*d*) $2\cdot\dfrac{1}{\sqrt{7}}+3\sqrt{28}-\sqrt{63}=\frac{2}{7}\sqrt{7}+6\sqrt{7}-3\sqrt{7}=\frac{23}{7}\sqrt{7}$

(*e*) $\sqrt[4]{64}-5\sqrt[6]{\frac{1}{8}}=2\sqrt{2}-\frac{5}{2}\sqrt{2}=-\frac{1}{2}\sqrt{2}$

(*f*) $\sqrt{3}\cdot\sqrt{15}=\sqrt{45}=3\sqrt{5}$

(*g*) $\sqrt[3]{18}\cdot\sqrt[3]{4}=\sqrt[3]{72}=2\sqrt[3]{9}$

(*h*) $(2\sqrt{5}+3)(3\sqrt{5}-4)=30-8\sqrt{5}+9\sqrt{5}-12=18+\sqrt{5}$

(*i*) $(2\sqrt{3}-3\sqrt{2})(5\sqrt{3}+\sqrt{2})=30+2\sqrt{6}-15\sqrt{6}-6=24-13\sqrt{6}$

(*j*) $i^5=i^4\cdot i=1\cdot i=i$

1.21 Simplify each of the following:

(*a*) $\dfrac{3+5\sqrt{2}}{2\sqrt{2}}=\dfrac{(3+5\sqrt{2})\sqrt{2}}{(2\sqrt{2})\sqrt{2}}=\dfrac{3\sqrt{2}+10}{4}$

(*b*) $\dfrac{4}{\sqrt{2}+\sqrt{3}}=\dfrac{4}{\sqrt{2}+\sqrt{3}}\cdot\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}=\dfrac{4\sqrt{2}-4\sqrt{3}}{2-3}=4\sqrt{3}-4\sqrt{2}$

(*c*) $\dfrac{2\sqrt{3}+3\sqrt{2}}{3\sqrt{5}-5\sqrt{3}}=\dfrac{2\sqrt{3}+3\sqrt{2}}{3\sqrt{5}-5\sqrt{3}}\cdot\dfrac{3\sqrt{5}+5\sqrt{3}}{3\sqrt{5}+5\sqrt{3}}=\dfrac{6\sqrt{15}+30+9\sqrt{10}+15\sqrt{6}}{45-75}$

$$=\frac{30+15\sqrt{6}+9\sqrt{10}+6\sqrt{15}}{-30}=-\frac{10+5\sqrt{6}+3\sqrt{10}+2\sqrt{15}}{10}$$

(*d*) $\dfrac{1+4\sqrt[3]{2}}{\sqrt[3]{2}}=\dfrac{(1+4\sqrt[3]{2})\sqrt[3]{4}}{\sqrt[3]{2}\cdot\sqrt[3]{4}}=\dfrac{\sqrt[3]{4}+4\sqrt[3]{8}}{\sqrt[3]{8}}=\dfrac{\sqrt[3]{4}+8}{2}$

1.22 The numerical value of a real number N ($|N|$) is defined as follows:

$$|N|=N \quad\text{if}\quad N>0; \qquad |N|=0 \quad\text{if}\quad N=0; \qquad |N|=-N \quad\text{if}\quad N<0.$$

Arrange each set of numbers so that they may be separated by $<$. (*a*) $|-12|, |\frac{5}{2}|, |-9|, |1|, |-2|, |6|$; (*b*) $|3+4|, |9-6|, |2-8|, |-3-6|$.

(*a*) $|1|<|-2|<|\frac{5}{2}|<|6|<|-9|<|-12|$ since $1<2<\frac{5}{2}<6<9<12$.

(*b*) $|9-6|<|2-8|<|3+4|<|-3-6|$ since $3<6<7<9$.

1.23 Find the directed distance AB, given (*a*) $A(2)$, $B(6)$; (*b*) $A(3)$, $B(-7)$; (*c*) $A(-2)$, $B(-8)$; (*d*) $A(-10)$, $B(2)$; (*e*) $A(-9)$, $B(-2)$; (*f*) $A(1)$, $B(x)$; (*g*) $A(x_1)$, $B(3)$; (*h*) $A(x_1)$, $B(x_2)$.

(*a*) $AB=6-2=4$ (*c*) $AB=-8-(-2)=-6$ (*e*) $AB=-2-(-9)=7$ (*g*) $AB=3-x_1$

(*b*) $AB=-7-3=-10$ (*d*) $AB=2-(-10)=12$ (*f*) $AB=x-1$ (*h*) $AB=x_2-x_1$

1.24 (*a*) On a number scale locate the points $A(-5)$, $B(1)$, $C(7)$ and show that $AB+BC+CA=0$. (*b*) Relabel the above points, reading from left to right, $B(-5)$, $C(1)$, $A(7)$ and show that $AB+BC+CA=0$.

(*a*) $AB+BC+CA=[1-(-5)]+(7-1)+(-5-7)=6+6-12=0$

(*b*) $AB+BC+CA=(-5-7)+[1-(-5)]+(7-1)=0$

1.25 Find the coordinate of the midpoint of the segments AB in Problem 1.23.

(*a*) $\frac{1}{2}(2+6)=4$ (*c*) $\frac{1}{2}[-2+(-8)]=-5$ (*e*) $\frac{1}{2}(-9-2)=-\frac{11}{2}$ (*g*) $\frac{1}{2}(x_1+3)$

(*b*) $\frac{1}{2}[3+(-7)]=-2$ (*d*) $\frac{1}{2}(-10+2)=-4$ (*f*) $\frac{1}{2}(1+x)$ (*h*) $\frac{1}{2}(x_1+x_2)$

1.26 Rewrite each of the following, using i:

(*a*) $\sqrt{-5} = i\sqrt{5}$ (*b*) $\sqrt{-4} = 2i$ (*c*) $\sqrt{-a^2} = ai$ (*d*) $\sqrt{-32} = 4i\sqrt{2}$

(*e*) $3 - \sqrt{-9} = 3 - 3i$ (*f*) $\dfrac{8 + \sqrt{-16}}{2} = \dfrac{8 + 4i}{2} = 4 + 2i$

(*g*) $\dfrac{6 - \sqrt{-128}}{12} = \dfrac{6 - 8i\sqrt{2}}{12} = \dfrac{1}{2} - \dfrac{2\sqrt{2}}{3}i$

1.27 Perform the indicated operations.

(*a*) $(2 - 5i) + (4 + 3i) = (2 + 4) + (-5 + 3)i = 6 - 2i$

(*b*) $(3 + 2i) - (-6 - 3i) = [3 - (-6)] + [2 - (-3)]i = 9 + 5i$

(*c*) $(-5 + 2\sqrt{-4}) + (1 - \sqrt{-9}) = (-5 + 4i) + (1 - 3i) = -4 + i$

(*d*) $(2 + \sqrt{-27}) - (4 - \sqrt{-3}) = (2 + 3i\sqrt{3}) - (4 - i\sqrt{3}) = -2 + 4i\sqrt{3}$

(*e*) $(-2 - \sqrt{-8}) - (5 + \sqrt{-27}) = (-2 - 2i\sqrt{2}) - (5 + 3i\sqrt{3}) = -7 - (2\sqrt{2} + 3\sqrt{3})i$

(*f*) $(2 + 3i) + (2 - 3i) = 4$

(*g*) $(2 + 3i) - (2 - 3i) = 6i$

(*h*) $(2 - 5i)(4 + 3i) = 8 + 6i - 20i - 15i^2 = 8 - 14i + 15 = 23 - 14i$

(*i*) $(2 + 3i)(2 - 3i) = 4 - 6i + 6i - 9i^2 = 13$

Supplementary Problems

1.28 Arrange each of the following so that they may be separated by <.

(*a*) $\frac{2}{3}, -\frac{3}{4}, \frac{5}{6}, -1, \frac{4}{5}, -\frac{4}{3}, -\frac{1}{4}$ (*b*) $\frac{3}{2}, 2, \frac{7}{5}, \frac{4}{3}, 3$ (*c*) $\frac{3}{2}, \sqrt{3}, -\frac{1}{2}, -\sqrt{5}, 0$

1.29 Determine the greater of each pair.

(*a*) $|4 + (-2)|$ and $|-4| + |-2|$ (*b*) $|4 + (-2)|$ and $|4| + |-2|$ (*c*) $|4 - (-2)|$ and $|4| - |-2|$

1.30 Convert each of the following fractions into equivalent fractions having the indicated denominator:

(*a*) $\frac{3}{5}$, 15 (*b*) $-\frac{3}{5}$, 20 (*c*) $\frac{7}{3}$, 42 (*d*) $\frac{5}{7}$, 35 (*e*) $\frac{12}{13}$, 156

1.31 Perform the indicated operations.

(*a*) $(-2)(3)(-5)$

(*b*) $3(-2)(4) + (-5)(2)(0)$

(*c*) $-8 - (-6) + 2$

(*d*) $\frac{3}{4} + \frac{2}{3}$

(*e*) $\frac{3}{4} - \frac{2}{3}$

(*f*) $\frac{5}{6} - \frac{1}{2} - \frac{2}{3}$

(*g*) $\frac{3}{4} - \frac{7}{12} - \frac{1}{3}$

(*h*) $(\frac{1}{2})(\frac{8}{9})(\frac{6}{5})$

(*i*) $\frac{3}{8} \times 5\frac{1}{3}$

(*j*) $2\frac{1}{4} \times 2\frac{2}{3} \times 1\frac{2}{5} \times 2\frac{1}{7}$

(*k*) $\frac{25}{32} \div \frac{35}{64}$

(*l*) $3\frac{1}{3} \div \frac{7}{10}$

(*m*) $(1\frac{1}{2} \times 2\frac{1}{4}) \div 1\frac{1}{8}$

(*n*) $\dfrac{3 - \frac{2}{3}}{5 + \frac{5}{6}}$

(*o*) $\dfrac{\frac{2}{3} + \frac{3}{4}}{\frac{5}{6} - \frac{7}{8}}$

(*p*) $\dfrac{1\frac{1}{2} - 2\frac{2}{3}}{3\frac{1}{5} - 1\frac{1}{4}}$

1.32 Perform the indicated operations.

(*a*) $5\sqrt{3} + 2\sqrt{3} - 8\sqrt{3}$

(*b*) $5\sqrt{2} + \sqrt{32} - 3\sqrt{8}$

(*c*) $\sqrt[3]{12} \cdot \sqrt[3]{36}$

(*d*) $(1 + \sqrt{2})(3 - \sqrt{2})$

(*e*) $(2\sqrt{3} + 3\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$

(*f*) $(4\sqrt{3} - 3\sqrt{5})(2\sqrt{3} + \sqrt{5})$

(*g*) $\dfrac{4 - 2\sqrt{3}}{5\sqrt{3}}$

(*h*) $\dfrac{2\sqrt{5} - 3\sqrt{2}}{3\sqrt{5} + 4\sqrt{2}}$

(*i*) $\dfrac{3\sqrt{2} - 4\sqrt{3}}{4\sqrt{2} - 3\sqrt{3}}$