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Representations of Shifted Yangians and Finite W-algebras

Abstract

We study highest weight representations of shifted Yangians over an algebraically closed field of characteristic 0. In particular, we classify the finite dimensional irreducible representations and explain how to compute their Gelfand-Tsetlin characters in terms of known characters of standard modules and certain Kazhdan-Lusztig polynomials. Our approach exploits the relationship between shifted Yangians and the finite W-algebras associated to nilpotent orbits in general linear Lie algebras.

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Contents

CHAPTER 1

Introduction

Following work of Premet, there has been renewed interest recently in the representation theory of certain algebras that are associated to nilpotent orbits in complex semisimple Lie algebras. We refer to these algebras as *finite W-algebras*. They should be viewed as analogues of universal enveloping algebras for the Slodowy slice through the nilpotent orbit in question. Actually, in the special cases considered in this article, the definition of these algebras first appeared in 1979 in the Ph.D. thesis of Lynch [**Ly**], extending the celebrated work of Kostant [**Ko2**] treating regular nilpotent orbits. However, despite quite a lot of attention by a number of authors since then, see e.g. [**Ka, M, Ma, BT, VD, GG, P1, P2, DK**], there is still surprisingly little concrete information about the representation theory of these algebras to be found in the literature. The goal in this article is to undertake a thorough study of finite dimensional representations of the finite W-algebras associated to nilpotent orbits in the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$. We are able to make progress in this case thanks largely to the relationship between finite W-algebras and *shifted Yangians* first noticed in [**RS, BR**] and developed in full generality in [**BK5**].

Fix for the remainder of the introduction a partition $\lambda = (p_1 \leq \cdots \leq p_n)$ of N. We draw the Young diagram of λ in a slightly unconventional way, so that there are p_i boxes in the ith row, numbering rows $1, \dots, n$ from top to bottom in order of increasing length. Also number the non-empty columns of this diagram by $1, \dots, l$ from left to right, and let q_i denote the number of boxes in the ith column, so $\lambda' = (q_1 \geq \cdots \geq q_l)$ is the transpose partition to λ. For example, if $(p_1, p_2, p_3) = (2, 3, 4)$ then the Young diagram of λ is

1	4		
2	5	7	
3	6	8	9

and $(q_1, q_2, q_3, q_4) = (3, 3, 2, 1)$. We number the boxes of the diagram by $1, 2, \dots, N$ down columns from left to right, and let $\mathrm{row}(i)$ and $\mathrm{col}(i)$ denote the row and column numbers of the ith box.

Writing $e_{i,j}$ for the ij-matrix unit in the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$, let e denote the matrix $\sum_{i,j} e_{i,j}$ summing over all $1 \leq i, j \leq N$ such that $\mathrm{row}(i) = \mathrm{row}(j)$ and $\mathrm{col}(i) = \mathrm{col}(j) - 1$. This is a nilpotent matrix of Jordan type λ. For instance, if λ is as above, then $e = e_{1,4} + e_{2,5} + e_{5,7} + e_{3,6} + e_{6,8} + e_{8,9}$. Define a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ of the Lie algebra $\mathfrak{g}$ by declaring that each $e_{i,j}$ is of degree $(\mathrm{col}(j) - \mathrm{col}(i))$. This is a *good grading* for $e \in \mathfrak{g}_1$ in the sense of [**KRW**] (see also [**EK**] for the full classification). However, it is not the usual Dynkin grading arising from an $\mathfrak{sl}_2$-triple unless all the parts of λ are equal. Actually, in the main body of the article, we work with more general good gradings than the one described here, replacing the Young diagram of λ with a more general diagram called a *pyramid* and denoted by the symbol π; see §3.1. When the pyramid π is left-justified, it coincides with the Young diagram of λ. We have chosen to focus just on this case in the introduction, since it plays a distinguished role in the theory.

Now we give a formal definition of the finite W-algebra $W(\lambda)$ associated to this data. Let $\mathfrak{p}$ denote the parabolic subalgebra $\bigoplus_{j\geq 0}\mathfrak{g}_j$ of $\mathfrak{g}$ with Levi factor $\mathfrak{h} = \mathfrak{g}_0$, and let $\mathfrak{m}$ denote the opposite nilradical $\bigoplus_{j<0}\mathfrak{g}_j$. Taking the trace form with e defines a one dimensional representation $\chi : \mathfrak{m} \to \mathbb{C}$. Let I_χ be the two-sided ideal of the universal enveloping algebra $U(\mathfrak{m})$ generated by $\ker\chi$. Let $\eta : U(\mathfrak{p}) \to U(\mathfrak{p})$ be the automorphism mapping $e_{i,j} \mapsto e_{i,j} + \delta_{i,j}(n - q_{\mathrm{col}(j)} - q_{\mathrm{col}(j)+1} - \cdots - q_l)$ for each $e_{i,j} \in \mathfrak{p}$. Then, by our definition, $W(\lambda)$ is the following subalgebra of $U(\mathfrak{p})$:

$$W(\lambda) = \{u \in U(\mathfrak{p}) \mid [x, \eta(u)] \in U(\mathfrak{g})I_\chi \text{ for all } x \in \mathfrak{m}\};$$

see §3.2. The twist by the automorphism η here is unconventional but quite convenient later on; it is analogous to "shifting by ρ" in the definition of Harish-Chandra homomorphism. For examples, if the Young diagram of λ consists of a single column and e is the zero matrix, $W(\lambda)$ coincides with the entire universal enveloping algebra $U(\mathfrak{g})$. At the other extreme, if the Young diagram of λ consists of a single row and e is a regular nilpotent element, the work of Kostant [**Ko2**] shows that $W(\lambda)$ is isomorphic to the center of $U(\mathfrak{g})$, in particular it is commutative.

For $u \in W(\lambda)$, right multiplication by $\eta(u)$ leaves $U(\mathfrak{g})I_\chi$ invariant, so induces a well-defined right action of u on the *generalized Gelfand-Graev representation*

$$Q_\chi = U(\mathfrak{g})/U(\mathfrak{g})I_\chi \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi.$$

This makes Q_χ into a $(U(\mathfrak{g}), W(\lambda))$-bimodule. The associated algebra homomorphism $W(\lambda) \to \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{\mathrm{op}}$ is actually an isomorphism, giving an alternate definition of $W(\lambda)$ as an endomorphism algebra.

Another useful construction involves the homomorphism $\xi : U(\mathfrak{p}) \to U(\mathfrak{h})$ induced by the natural projection $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$. The restriction of ξ to $W(\lambda)$ defines an *injective* algebra homomorphism $W(\lambda) \hookrightarrow U(\mathfrak{h})$ which we call the *Miura transform*; see §3.6. To explain its signigicance, we note that $\mathfrak{h} = \mathfrak{gl}_{q_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{q_l}(\mathbb{C})$, so $U(\mathfrak{h})$ is naturally identified with the tensor product $U(\mathfrak{gl}_{q_1}(\mathbb{C})) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l}(\mathbb{C}))$. Given $\mathfrak{gl}_{q_i}(\mathbb{C})$-modules M_i for each $i = 1, \dots, l$, the outer tensor product $M_1 \boxtimes \cdots \boxtimes M_l$ is therefore a $U(\mathfrak{h})$-module in the natural way. Hence, restricting via the Miura transform, $M_1 \boxtimes \cdots \boxtimes M_l$ is a $W(\lambda)$-module too. This construction plays the role of tensor product in the representation theory of $W(\lambda)$.

Next we want to recall the connection between $W(\lambda)$ and shifted Yangians. Let σ be the upper triangular $n \times n$ matrix with ij-entry $(p_j - p_i)$ for $i \leq j$. The *shifted Yangian* $Y_n(\sigma)$ associated to σ is the associative algebra over $\mathbb{C}$ with generators $D_i^{(r)}$ $(1 \leq i \leq n, r > 0)$, $E_i^{(r)}$ $(1 \leq i < n, r > p_{i+1} - p_i)$ and $F_i^{(r)}$ $(1 \leq i < n, r > 0)$ subject to certain relations recorded explicitly in §2.1. In the case that σ is the zero matrix, i.e. all parts of λ are equal, $Y_n(\sigma)$ is precisely the usual Yangian Y_n associated to the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ and the defining relations are a variation on the Drinfeld presentation of [**D**]; see [**BK4**]. In general, the presentation of $Y_n(\sigma)$ is adapted to its natural *triangular decomposition*, allowing us to study representations in terms of highest weight theory. In particular, the subalgebra generated by all the elements $D_i^{(r)}$ is a maximal commutative subalgebra which we call the *Gelfand-Tsetlin subalgebra*. We often work with the generating functions

$$D_i(u) = 1 + D_i^{(1)}u^{-1} + D_i^{(2)}u^{-2} + \cdots \in Y_n(\sigma)[[u^{-1}]].$$

The main result of [**BK5**] shows that the finite W-algebra $W(\lambda)$ is isomorphic to the quotient of $Y_n(\sigma)$ by the two-sided ideal generated by all $D_1^{(r)}$ $(r > p_1)$. The precise identification of $W(\lambda)$ with this quotient is described in §3.4. Also in §3.6, we explain how the tensor product construction outlined in the previous paragraph is induced by the comultiplication of the Hopf algebra Y_n.

We are ready to describe the first results about representation theory. We call a vector v in a $Y_n(\sigma)$-module M a *highest weight vector* if it is annihilated by all $E_i^{(r)}$ and each $D_i^{(r)}$ acts on v by a scalar. A critical point is that if v is a highest weight vector in a $W(\lambda)$-module, viewed as a $Y_n(\sigma)$-module via the map $Y_n(\sigma) \twoheadrightarrow W(\lambda)$, then in fact $D_i^{(r)}v = 0$ for all $r > p_i$. This is obvious for $i = 1$, since the image of $D_1^{(r)}$ in $W(\lambda)$ is zero by the definition of the map for all $r > p_1$. For $i > 1$, it follows from the following fundamental result proved in §3.7: for any i and $r > p_i$, the image of $D_i^{(r)}$ in $W(\lambda)$ is congruent to zero modulo the left ideal generated by all $E_j^{(s)}$. Hence, if v is a highest weight vector in a $W(\lambda)$-module, then there exist scalars $(a_{i,j})_{1\leq i\leq n, 1\leq j\leq p_i}$ such that

$$\begin{aligned}
u^{p_1} D_1(u)v &= (u + a_{1,1})(u + a_{1,2}) \cdots (u + a_{1,p_1})v, \\
(u-1)^{p_2} D_2(u-1)v &= (u + a_{2,1})(u + a_{2,2}) \cdots (u + a_{2,p_2})v, \\
&\ \ \vdots \\
(u-n+1)^{p_n} D_n(u-n+1)v &= (u + a_{n,1})(u + a_{n,2}) \cdots (u + a_{n,p_n})v.
\end{aligned}$$

Let A be the λ-tableau obtained by writing the scalars $a_{i,1}, \dots, a_{i,p_i}$ into the boxes on the ith row of the Young diagram of λ. In this way, the highest weights that can arise in $W(\lambda)$-modules are parametrized by the set $\mathrm{Row}(\lambda)$ of *row symmetrized λ-tableaux*, i.e. tableaux of shape λ with entries from $\mathbb{C}$ viewed up to row equivalence. Conversely, given any row symmetrized λ-tableau $A \in \mathrm{Row}(\lambda)$, there exists a (non-zero) universal highest weight module $M(A)$ generated by such a highest weight vector; see §6.1. We call $M(A)$ the *generalized Verma module* of type A. By familiar arguments, $M(A)$ has a unique irreducible quotient $L(A)$, and then the modules $\{L(A) \mid A \in \mathrm{Row}(\lambda)\}$ give all irreducible highest weight modules for $W(\lambda)$ up to isomorphism.

There is a natural abelian category $\mathcal{M}(\lambda)$ which is an analogue of the BGG category $\mathcal{O}$ for the algebra $W(\lambda)$; see §7.5. (Actually, $\mathcal{M}(\lambda)$ is more like the category $\mathcal{O}^\infty$ obtained by weakening the hypothesis that a Cartan subalgebra acts semisimply in the usual definition of $\mathcal{O}$.) All objects in $\mathcal{M}(\lambda)$ are of finite length, and the simple objects are precisely the irreducible highest weight modules, hence the isomorphism classes $\{[L(A)] \mid A \in \mathrm{Row}(\lambda)\}$ give a canonical basis for the Grothendieck group $[\mathcal{M}(\lambda)]$ of the category $\mathcal{M}(\lambda)$. The generalized Verma modules belong to $\mathcal{M}(\lambda)$ too, and it is natural to consider the *composition multiplicities* $[M(A) : L(B)]$ for $A, B \in \mathrm{Row}(\lambda)$. We will formulate a precise combinatorial conjecture for these, in the spirit of the Kazhdan-Lusztig conjecture, later on in the introduction. For now, we just record the following basic result about the structure of Verma modules; see §6.3. For the statement, let $\leq$ denote the Bruhat ordering on row symmetrized λ-tableaux; see §4.1.

THEOREM A **(Linkage principle).** *For $A, B \in \mathrm{Row}(\lambda)$, the composition multiplicity $[M(A) : L(A)]$ is equal to 1, and $[M(A) : L(B)] \neq 0$ if and only if $B \leq A$ in the Bruhat ordering.*

Hence, $\{[M(A)] \mid A \in \mathrm{Row}(\lambda)\}$ is another natural basis for the Grothendieck group $[\mathcal{M}(\lambda)]$. We want to say a few words about the proof of Theorem A, since it involves an interesting technique. Modules in the category $\mathcal{M}(\lambda)$ possess *Gelfand-Tsetlin characters*; see §5.2. This is a formal notion that keeps track of the dimensions of the generalized weight space decomposition of a module with respect to the Gelfand-Tsetlin subalgebra of $Y_n(\sigma)$, similar in spirit to the q-characters of Frenkel and Reshetikhin [**FR**]. The map sending a module to its Gelfand-Tsetlin character induces an embedding of the Grothendieck group $[\mathcal{M}(\lambda)]$ into a certain completion of the ring of Laurent polynomials $\mathbb{Z}[y_{i,a}^{\pm 1} \mid i = 1, \dots, n, a \in \mathbb{C}]$, for indeterminates $y_{i,a}$. The key step in our proof of Theorem A is the computation of the Gelfand-Tsetlin character of the Verma module $M(A)$ itself; see §6.2 for the precise statement. In general, $\mathrm{ch}\, M(A)$ is an infinite sum of monomials in the $y_{i,a}^{\pm 1}$'s involving both positive and negative powers, but the highest weight vector of $M(A)$ contributes just the positive monomial

$$y_{1,a_{1,1}} \cdots y_{1,a_{1,p_1}} \times y_{2,a_{2,1}} \cdots y_{2,a_{2,p_2}} \times \cdots \times y_{n,a_{n,1}} \cdots y_{n,a_{n,p_n}},$$

where $a_{i,1}, \dots, a_{i,p_i}$ are the entries in the ith row of A as above. The highest weight vector of any composition factor contributes a similar such positive monomial. So by analyzing the positive monomials appearing in the formula for $\mathrm{ch}\, M(A)$, we get information about the possible $L(B)$'s that can be composition factors of $M(A)$. The Bruhat ordering on tableaux emerges naturally out of these considerations.

Another important property of Verma modules has to do with tensor products. Let $A \in \mathrm{Row}(\lambda)$ be a row symmetrized λ-tableau. Pick any representative for it and let A_i denote the ith column of this representative with entries $a_{i,1}, \dots, a_{i,q_i}$ read from top to bottom. Let $M(A_i)$ denote the usual Verma module for the Lie algebra $\mathfrak{gl}_{q_i}(\mathbb{C})$ generated by a highest weight vector v_+ annihilated by all strictly upper triangular matrices and on which $e_{j,j}$ acts as the scalar $(a_{i,j} + n - q_i + j - 1)$ for each $j = 1, \dots, q_i$. Via the Miura transform, the tensor product $M(A_1) \boxtimes \cdots \boxtimes M(A_l)$ is then naturally a $W(\lambda)$-module as explained above, and the vector $v_+ \otimes \cdots \otimes v_+$ is a highest weight vector of type A in this tensor product. In fact, our formula for the Gelfand-Tsetlin character of $M(A)$ implies that

$$[M(A)] = [M(A_1) \boxtimes \cdots \boxtimes M(A_l)],$$

equality in the Grothendieck group $[\mathcal{M}(\lambda)]$. The first part of the next theorem, proved in §6.4, is a consequence of this equality; the second part is an application of [**FO**].

THEOREM B **(Structure of center).** *Identifying $W(\lambda)$ with the endomorphism algebra of Q_χ, the natural multiplication map $\psi : Z(U(\mathfrak{g})) \to \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)$ defines an algebra isomorphism between the center of $U(\mathfrak{g})$ and the center of $W(\lambda)$. Moreover, $W(\lambda)$ is free as a module over its center.*

Now we switch our attention to finite dimensional $W(\lambda)$-modules. Let $\mathcal{F}(\lambda)$ denote the category of all finite dimensional $W(\lambda)$-module, viewed as a subcategory of the category $\mathcal{M}(\lambda)$. The problem of classifying all finite dimensional irreducible

$W(\lambda)$-modules reduces to determining precisely which $A \in \mathrm{Row}(\lambda)$ have the property that $L(A)$ is finite dimensional. To formulate the final result, we need one more definition. Call a λ-tableau A with entries in $\mathbb{C}$ *column strict* if in every column the entries belong to the same coset of $\mathbb{C}$ modulo $\mathbb{Z}$ and are strictly increasing from bottom to top. Let $\mathrm{Col}(\lambda)$ denote the set of all such column strict λ-tableaux. There is an obvious map

$$R : \mathrm{Col}(\lambda) \to \mathrm{Row}(\lambda)$$

mapping a λ-tableau to its row equivalence class. Let $\mathrm{Dom}(\lambda)$ denote the image of this map, the set of all *dominant* row symmetrized λ-tableaux.

THEOREM C **(Finite dimensional irreducible representations).** *For $A \in \mathrm{Row}(\lambda)$, the irreducible highest weight module $L(A)$ is finite dimensional if and only if A is dominant. Hence, the modules $\{L(A) \mid A \in \mathrm{Dom}(\lambda)\}$ form a complete set of pairwise non-isomorphic finite dimensional irreducible $W(\lambda)$-modules.*

To prove this, there are two steps: one needs to show first that each $L(A)$ with $A \in \mathrm{Dom}(\lambda)$ is finite dimensional, and second that all other $L(A)$'s are infinite dimensional. Let us explain the argument for the first step. Given $A \in \mathrm{Col}(\lambda)$, let A_i be its ith column and define $L(A_i)$ to be the unique irreducible quotient of the Verma module $M(A_i)$ introduced above. Because A is column strict, each $L(A_i)$ is a finite dimensional irreducible $\mathfrak{gl}_{q_i}(\mathbb{C})$-module. Hence we obtain a finite dimensional $W(\lambda)$-module

$$V(A) = L(A_1) \boxtimes \cdots \boxtimes L(A_l),$$

which we call the *standard module* corresponding to $A \in \mathrm{Col}(\lambda)$. It contains an obvious highest weight vector of type equal to the row equivalence class of A. This simple construction is enough to finish the first step of the proof. The second step is actually much harder, and is an extension of the proof due to Tarasov [**T1, T2**] and Drinfeld [**D**] of the classification of finite dimensional irreducible representations of the Yangian Y_n by *Drinfeld polynomials.* It is based on the remarkable fact that when $n = 2$, i.e. the Young diagram of λ has just two rows, *every* $L(A) (A \in \mathrm{Row}(\lambda))$ can be expressed as an irreducible tensor product; see §7.1.

Amongst all the standard modules, there are some special ones which are highest weight modules and whose isomorphism classes form a basis for the Grothendieck group of the category $\mathcal{F}(\lambda)$. Let $A \in \mathrm{Col}(\lambda)$ be a column strict λ-tableau with entries $a_{i,1}, \dots, a_{i,p_i}$ in its ith row read from left to right. We say that A is *standard* if $a_{i,j} \leq a_{i,k}$ for every $1 \leq i \leq n$ and $1 \leq j < k \leq p_i$ such that $a_{i,j}$ and $a_{i,k}$ belong to the same coset of $\mathbb{C}$ modulo $\mathbb{Z}$. If all entries of A are integers, this is the usual definition of a standard tableau: entries increase strictly up columns and weakly along rows. Let $\mathrm{Std}(\lambda)$ denote the set of all standard λ-tableaux $A \in \mathrm{Col}(\lambda)$. Our proof of the next theorem is based on an argument due to Chari [**C**] in the context of quantum affine algebras; see §7.3.

THEOREM D **(Highest weight standard modules).** *For $A \in \mathrm{Std}(\lambda)$, the standard module $V(A)$ is a highest weight module of highest weight equal to the row equivalence class of A.*

Most of the results so far are analogous to well known results in the representation theory of Yangians and quantum affine algebras, and do not exploit the finite W-algebra side of the picture in any significant way. To remedy this, we need to apply *Skryabin's theorem* from [**Sk**]; see §8.1. This asserts that the functor

$Q_\chi \otimes_{W(\lambda)} ?$ gives an equivalence of categories between the category of all $W(\lambda)$-modules and the category $\mathcal{W}(\lambda)$ of all *generalized Whittaker modules*, namely, all $\mathfrak{g}$-modules on which $(x - \chi(x))$ acts locally nilpotently for all $x \in \mathfrak{m}$. For any finite dimensional $\mathfrak{g}$-module V, it is obvious that the functor $? \otimes V$ maps objects in $\mathcal{W}(\lambda)$ to objects in $\mathcal{W}(\lambda)$. Transporting through Skryabin's equivalence of categories, we obtain a functor $? \circledast V$ on $W(\lambda)$-mod itself; see §8.2. In this way, one can introduce *translation functors* on the categories $\mathcal{M}(\lambda)$ and $\mathcal{F}(\lambda)$. Actually, we just need some special translation functors, peculiar to the type A theory and denoted e_i, f_i for $i \in \mathbb{C}$, which arise from $\circledast$'ing with the natural $\mathfrak{gl}_N(\mathbb{C})$-module and its dual. These functors fit into the axiomatic framework developed recently by Chuang and Rouquier [**CR**]; see §8.3.

Now recall the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{h}$. We let $\mathcal{O}(\lambda)$ denote the corresponding parabolic category $\mathcal{O}$, the category of all finitely generated $\mathfrak{g}$-modules on which $\mathfrak{p}$ acts locally finitely and $\mathfrak{h}$ acts semisimply. For $A \in \mathrm{Col}(\lambda)$ with entry a_i in its ith box, we let $N(A) \in \mathcal{O}(\lambda)$ denote the *parabolic Verma module* generated by a highest weight vector v_+ that is annihilated by all strictly upper triangular matrices in $\mathfrak{g}$ and on which $e_{i,i}$ acts as the scalar $(a_i + i - 1)$ for each $i = 1, \dots, N$. Let $K(A)$ denote the unique irreducible quotient of $N(A)$. Both of the sets $\{[N(A)] \mid A \in \mathrm{Col}(\lambda)\}$ and $\{[K(A)] \mid A \in \mathrm{Col}(\lambda)\}$ form natural bases for the Grothendieck group $[\mathcal{O}(\lambda)]$. There is a remarkable functor

$$\mathbb{V} : \mathcal{O}(\lambda) \to \mathcal{F}(\lambda)$$

introduced originally (in a slightly different form) by Kostant and Lynch. We call it the *Whittaker functor*; see §8.5. It is an exact functor preserving central characters and commuting with translation functors. Moreover, it maps the parabolic Verma module $N(A)$ to the standard module $V(A)$ for every $A \in \mathrm{Col}(\lambda)$. The culmination of this article is the following theorem.

THEOREM E (**Construction of irreducible modules**). *The Whittaker functor $\mathbb{V} : \mathcal{O}(\lambda) \to \mathcal{F}(\lambda)$ sends irreducible modules to irreducible modules or zero. More precisely, take any $A \in \mathrm{Col}(\lambda)$ and let $B \in \mathrm{Row}(\lambda)$ be its row equivalence class. Then*

$$\mathbb{V}(K(A)) \cong \begin{cases} L(B) & \text{if } A \text{ is standard,} \\ 0 & \text{otherwise.} \end{cases}$$

Every finite dimensional irreducible $W(\lambda)$-module arises in this way.

There are three main ingredients to the proof of this theorem. First, we need detailed information about the translation functors e_i, f_i, much of which is provided by [**CR**] as an application of the representation theory of degenerate affine Hecke algebras. Second, we need to know that the standard modules $V(A)$ have simple cosocle if $A \in \mathrm{Std}(\lambda)$, which follows from Theorem D. Finally, we need to apply the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$ in order to determine exactly when $\mathbb{V}(K(A))$ is non-zero.

Let us discuss some of the combinatorial consequences of Theorem E in more detail. For this, we at last restrict our attention just to modules having integral central character. Let $\mathrm{Row}_0(\lambda), \mathrm{Col}_0(\lambda), \mathrm{Dom}_0(\lambda)$ and $\mathrm{Std}_0(\lambda)$ denote the subsets of $\mathrm{Row}(\lambda), \mathrm{Col}(\lambda), \mathrm{Dom}(\lambda)$ and $\mathrm{Std}(\lambda)$ consisting of the tableaux all of whose entries are integers. The restriction of the map R actually gives a bijection between the sets $\mathrm{Std}_0(\lambda)$ and $\mathrm{Dom}_0(\lambda)$. Let $\mathcal{O}_0(\lambda)$, $\mathcal{F}_0(\lambda)$ and $\mathcal{M}_0(\lambda)$ denote the full subcategories of $\mathcal{O}(\lambda)$, $\mathcal{F}(\lambda)$ and $\mathcal{M}(\lambda)$ consiting of objects all of whose composition factors are of the form $\{K(A) \mid A \in \mathrm{Col}_0(\lambda)\}$, $\{L(A) \mid A \in \mathrm{Dom}_0(\lambda)\}$ and $\{L(A) \mid A \in \mathrm{Row}_0(\lambda)\}$, respectively. The isomorphism classes of these three sets of objects give canonical bases for the Grothendieck groups $[\mathcal{O}_0(\lambda)], [\mathcal{F}_0(\lambda)]$ and $[\mathcal{M}_0(\lambda)]$, as do the isomorphism classes of the parabolic Verma modules $\{N(A) \mid A \in \mathrm{Col}_0(\lambda)\}$, the standard modules $\{V(A) \mid A \in \mathrm{Std}_0(\lambda)\}$, and the generalized Verma modules $\{M(A) \mid A \in \mathrm{Row}_0(\lambda)\}$, respectively.

The functor $\mathbb{V}$ above restricts to an exact functor $\mathbb{V} : \mathcal{O}_0(\lambda) \to \mathcal{F}_0(\lambda)$, and we also have the natural embedding $\mathbb{I}$ of the category $\mathcal{F}_0(\lambda)$ into $\mathcal{M}_0(\lambda)$. At the level of Grothendieck groups, these functors induce maps

$$[\mathcal{O}_0(\lambda)] \overset{\mathbb{V}}{\twoheadrightarrow} [\mathcal{F}_0(\lambda)] \overset{\mathbb{I}}{\hookrightarrow} [\mathcal{M}_0(\lambda)].$$

The translation functors e_i, f_i for $i \in \mathbb{Z}$ (and more generally their divided powers $e_i^{(r)}, f_i^{(r)}$ defined as in [**CR**]) induce maps also denoted e_i, f_i on all these Grothendieck groups. The resulting maps satisfy the relations of the Chevalley generators (and their divided powers) for the Kostant $\mathbb{Z}$-form $U_\mathbb{Z}$ of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_\infty(\mathbb{C})$, that is, the Lie algebra of matrices with rows and columns labelled by $\mathbb{Z}$ all but finitely many of which are zero. The maps $\mathbb{V}$ and $\mathbb{I}$ are then $U_\mathbb{Z}$-module homomorphisms with respect to these actions.

Now the point is that all of this categorifies a well known situation in linear algebra. Let $V_\mathbb{Z}$ denote the natural $U_\mathbb{Z}$-module, with basis v_i $(i \in \mathbb{Z})$. We write $\bigwedge^{\lambda'}(V_\mathbb{Z})$ for the tensor product $\bigwedge^{q_1}(V_\mathbb{Z}) \otimes \cdots \otimes \bigwedge^{q_l}(V_\mathbb{Z})$ and $S^\lambda(V_\mathbb{Z})$ for the tensor product $S^{p_1}(V_\mathbb{Z}) \otimes \cdots \otimes S^{p_n}(V_\mathbb{Z})$. These free $\mathbb{Z}$-modules have natural monomial bases denoted $\{N_A \mid A \in \mathrm{Col}_0(\lambda)\}$ and $\{M_A \mid A \in \mathrm{Row}_0(\lambda)\}$, respectively; see §4.2. A well known consequence of the Littlewood-Richardson rule (observed already by Young long before) implies that the space

$$\mathrm{Hom}_{U_\mathbb{Z}}(\textstyle\bigwedge^{\lambda'}(V_\mathbb{Z}), S^\lambda(V_\mathbb{Z}))$$

is a free $\mathbb{Z}$-module of rank one; indeed, there is a canonical $U_\mathbb{Z}$-module homomorphism $\mathbb{V} : \bigwedge^{\lambda'}(V_\mathbb{Z}) \to S^\lambda(V_\mathbb{Z})$ that generates the space of all such maps. The image of this map is $P^\lambda(V_\mathbb{Z})$, a familiar $\mathbb{Z}$-form for the *irreducible polynomial representation* of $\mathfrak{gl}_\infty(\mathbb{C})$ labelled by the partition λ. So by definition $P^\lambda(V_\mathbb{Z})$ is a subspace of $S^\lambda(V_\mathbb{Z})$; we denote the natural inclusion by $\mathbb{I}$. Recall $P^\lambda(V_\mathbb{Z})$ also possesses a standard monomial basis $\{V_A \mid A \in \mathrm{Std}_0(\lambda)\}$, defined from $V_A = \mathbb{V}(N_A)$. Finally, we let $i : \bigwedge^{\lambda'}(V_\mathbb{Z}) \to [\mathcal{O}_0(\lambda)]$, $j : P^\lambda(V_\mathbb{Z}) \to [\mathcal{F}_0(\lambda)]$ and $k : S^\lambda(V_\mathbb{Z}) \to [\mathcal{M}_0(\lambda)]$ be the $\mathbb{Z}$-module homomorphisms sending $N_A \mapsto N(A), V_A \mapsto [V(A)]$ and $M_A \mapsto [M(A)]$ for $A \in \mathrm{Col}_0(\lambda)$, $A \in \mathrm{Std}_0(\lambda)$ and $A \in \mathrm{Row}_0(\lambda)$, respectively.

THEOREM F (**Categorification of polynomial functors**). *The maps i, j, k are all isomorphisms of $U_\mathbb{Z}$-modules, and the following diagram commutes:*

$$\begin{array}{ccccc} \bigwedge^{\lambda'}(V_\mathbb{Z}) & \xrightarrow{\mathbb{V}} & P^\lambda(V_\mathbb{Z}) & \xrightarrow{\mathbb{I}} & S^\lambda(V_\mathbb{Z}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\ [\mathcal{O}_0(\lambda)] & \xrightarrow[\mathbb{V}]{} & [\mathcal{F}_0(\lambda)] & \xrightarrow[\mathbb{I}]{} & [\mathcal{M}_0(\lambda)]. \end{array}$$

Moreover, setting $L_A = j^{-1}([L(A)])$ for $A \in \mathrm{Dom}_0(\lambda)$, the basis $\{L_A \mid A \in \mathrm{Dom}_0(\lambda)\}$ coincides with Lusztig's dual canonical basis/Kashiwara's upper global crystal basis for the polynomial representation $P^\lambda(V_\mathbb{Z})$.

Again, the Kazhdan-Lusztig conjecture plays the central role in the proof of this theorem. Actually, we use the following increasingly well known reformulation of the Kazhdan-Lusztig conjecture in type A: setting $K_A = i^{-1}([K(A)])$, the basis $\{K_A \mid A \in \mathrm{Col}_0(\lambda)\}$ coincides with the dual canonical basis for the space $\bigwedge^{\lambda'}(V_{\mathbb{Z}})$. In particular, this implies that the decomposition numbers $[V(A) : L(B)]$ for $A \in \mathrm{Std}_0(\lambda)$ and $B \in \mathrm{Dom}_0(\lambda)$ can be computed in terms of certain Kazhdan-Lusztig polynomials associated to the symmetric group S_N evaluated at $q = 1$. From a special case, one can also recover the analogous result for the Yangian Y_n itself. We mention this, because it is interesting to compare the strategy followed here with that of Arakawa [**A1**], who also computes the decomposition matrices of the Yangian in terms of Kazhdan-Lusztig polynomials starting from the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$, via [**AS**]. We speculate that there is also a geometric approach to the representation theory of shifted Yangians in the spirit of [**V**].

As promised earlier in the introduction, let us now formulate a precise conjecture that explains how to compute the decomposition numbers $[M(A) : L(B)]$ for all $A, B \in \mathrm{Row}_0(\lambda)$, also in terms of Kazhdan-Lusztig polynomials associated to the symmetric group S_N. Setting $L_A = k^{-1}([L(A)])$ for any $A \in \mathrm{Row}_0(\lambda)$, we conjecture that $\{L_A \mid A \in \mathrm{Row}_0(\lambda)\}$ coincides with the dual canonical basis for the space $S^\lambda(V_{\mathbb{Z}})$; see §7.5. This is a purely combinatorial reformulation in type A of the conjecture of de Vos and van Driel [**VD**] for arbitrary finite W-algebras, and is consistent with an idea of Premet that there should be an equivalence of categories between the category $\mathcal{M}(\lambda)$ here and a certain category $\mathcal{N}(\lambda)$ considered by Miličíc and Soergel [**MS**]. Our conjecture is known to be true in the special case that the Young diagram of λ consists of a single column: in that case it is precisely the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$. It is also true if the Young diagram of λ has at most two rows, as can be verified by comparing the explicit construction of the simple highest weight modules in the two row case from §7.1 with the explicit description of the dual canonical basis in this case from [**B**, Theorem 20]. Finally, Theorem E would be an easy consequence of this conjecture.

In a forthcoming article [**BK6**], we will study the categories of *polynomial* and *rational* representations of $W(\lambda)$ in more detail. In particular, we will make precise the relationship between polynomial representations of $W(\lambda)$ and representations of degenerate cyclotomic Hecke algebras, and we will relate the Whittaker functor $\mathbb{V}$ to work of Soergel [**S**] and Backelin [**Ba**]. This should have applications to the representation theory of *affine W-algebras* in the spirit of [**A2**].

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CHAPTER 2

Shifted Yangians

We will work from now on over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $\geq$ denote the partial order on $\mathbb{F}$ defined by $x \geq y$ if $(x - y) \in \mathbb{N}$, where $\mathbb{N}$ denotes $\{0, 1, 2, \dots\} \subset \mathbb{F}$. We write simply $\mathfrak{gl}_n$ for the Lie algebra $\mathfrak{gl}_n(\mathbb{F})$. In this preliminary chapter, we collect some basic definitions and results about shifted Yangians, most of which are taken from [**BK5**]. By a *shift matrix* we mean a matrix $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ of non-negative integers such that

$$s_{i,j} + s_{j,k} = s_{i,k} \tag{2.1}$$

whenever $|i-j|+|j-k| = |i-k|$. Note this means that $s_{1,1} = \cdots = s_{n,n} = 0$, and the matrix σ is completely determined by the upper diagonal entries $s_{1,2}, s_{2,3}, \dots, s_{n-1,n}$ and the lower diagonal entries $s_{2,1}, s_{3,2}, \dots, s_{n,n-1}$. We fix such a matrix σ throughout the chapter.

2.1. Generators and relations

The *shifted Yangian* associated to the matrix σ is the algebra $Y_n(\sigma)$ over $\mathbb{F}$ defined by generators

$$\{D_i^{(r)} \mid 1 \leq i \leq n, r > 0\}, \tag{2.2}$$

$$\{E_i^{(r)} \mid 1 \leq i < n, r > s_{i,i+1}\}, \tag{2.3}$$

$$\{F_i^{(r)} \mid 1 \leq i < n, r > s_{i+1,i}\} \tag{2.4}$$

subject to certain relations. In order to write down these relations, let

$$D_i(u) := \sum_{r \geq 0} D_i^{(r)} u^{-r} \in Y_n(\sigma)[[u^{-1}]] \tag{2.5}$$

where $D_i^{(0)} := 1$, and then define some new elements $\widetilde{D}_i^{(r)}$ of $Y_n(\sigma)$ from the equation

$$\widetilde{D}_i(u) = \sum_{r \geq 0} \widetilde{D}_i^{(r)} u^{-r} := -D_i(u)^{-1}. \tag{2.6}$$

With this notation, the relations are as follows.

$$[D_i^{(r)}, D_j^{(s)}] = 0, \tag{2.7}$$

$$[E_i^{(r)}, F_j^{(s)}] = \delta_{i,j} \sum_{t=0}^{r+s-1} \widetilde{D}_i^{(t)} D_{i+1}^{(r+s-1-t)}, \tag{2.8}$$

$$[D_i^{(r)}, E_j^{(s)}] = (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} D_i^{(t)} E_j^{(r+s-1-t)}, \tag{2.9}$$

$$[D_i^{(r)}, F_j^{(s)}] = (\delta_{i,j+1} - \delta_{i,j}) \sum_{t=0}^{r-1} F_j^{(r+s-1-t)} D_i^{(t)}, \tag{2.10}$$

$$[E_i^{(r)}, E_i^{(s+1)}] - [E_i^{(r+1)}, E_i^{(s)}] = E_i^{(r)} E_i^{(s)} + E_i^{(s)} E_i^{(r)}, \tag{2.11}$$

$$[F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] = F_i^{(r)} F_i^{(s)} + F_i^{(s)} F_i^{(r)}, \tag{2.12}$$

$$[E_i^{(r)}, E_{i+1}^{(s+1)}] - [E_i^{(r+1)}, E_{i+1}^{(s)}] = -E_i^{(r)} E_{i+1}^{(s)}, \tag{2.13}$$

$$[F_i^{(r+1)}, F_{i+1}^{(s)}] - [F_i^{(r)}, F_{i+1}^{(s+1)}] = -F_{i+1}^{(s)} F_i^{(r)}, \tag{2.14}$$

$$[E_i^{(r)}, E_j^{(s)}] = 0 \qquad \text{if } |i-j| > 1, \tag{2.15}$$

$$[F_i^{(r)}, F_j^{(s)}] = 0 \qquad \text{if } |i-j| > 1, \tag{2.16}$$

$$[E_i^{(r)}, [E_i^{(s)}, E_j^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_j^{(t)}]] = 0 \qquad \text{if } |i-j| = 1, \tag{2.17}$$

$$[F_i^{(r)}, [F_i^{(s)}, F_j^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_j^{(t)}]] = 0 \qquad \text{if } |i-j| = 1, \tag{2.18}$$

for all meaningful r, s, t, i, j. (For example, the relation (2.13) should be understood to hold for all $i = 1, \dots, n-2$, $r > s_{i,i+1}$ and $s > s_{i+1,i+2}$.)

It is often helpful to view $Y_n(\sigma)$ as an algebra graded by the root lattice Q_n associated to the Lie algebra $\mathfrak{gl}_n$. Let $\mathfrak{c}$ be the (abelian) Lie subalgebra of $Y_n(\sigma)$ spanned by the elements $D_1^{(1)}, \dots, D_n^{(1)}$. Let $\varepsilon_1, \dots, \varepsilon_n$ be the basis for $\mathfrak{c}^*$ dual to the basis $D_1^{(1)}, \dots, D_n^{(1)}$. We refer to elements of $\mathfrak{c}^*$ as *weights* and elements of

$$P_n := \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i \subset \mathfrak{c}^* \tag{2.19}$$

as *integral weights*. The *root lattice* associated to the Lie algebra $\mathfrak{gl}_n$ is then the $\mathbb{Z}$-submodule Q_n of P_n spanned by the *simple roots* $\varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, n-1$. We have the usual *dominance ordering* on $\mathfrak{c}^*$ defined by $\alpha \geq \beta$ if $(\alpha - \beta)$ is a sum of simple roots. With this notation set up, the relations imply that we can define a Q_n-grading

$$Y_n(\sigma) = \bigoplus_{\alpha \in Q_n} (Y_n(\sigma))_\alpha \tag{2.20}$$

of the algebra $Y_n(\sigma)$ by declaring that the generators $D_i^{(r)}, E_i^{(r)}$ and $F_i^{(r)}$ are of degrees $0, \varepsilon_i - \varepsilon_{i+1}$ and $\varepsilon_{i+1} - \varepsilon_i$, respectively.

2.2. PBW theorem

For $1 \leq i < j \leq n$ and $r > s_{i,j}$ resp. $r > s_{j,i}$, we inductively define the *higher root elements* $E_{i,j}^{(r)}$ resp. $F_{i,j}^{(r)}$ of $Y_n(\sigma)$ from the formulae

$$E_{i,i+1}^{(r)} := E_i^{(r)}, \qquad E_{i,j}^{(r)} := [E_{i,j-1}^{(r-s_{j-1,j})}, E_{j-1}^{(s_{j-1,j}+1)}], \tag{2.21}$$

$$F_{i,i+1}^{(r)} := F_i^{(r)}, \qquad F_{i,j}^{(r)} := [F_{j-1}^{(s_{j,j-1}+1)}, F_{i,j-1}^{(r-s_{j,j-1})}]. \tag{2.22}$$