Approximation Theory and Spline Functions

edited by S. P. Singh, J. H. W. Burry and B. Watson

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PREFACE

A NATO Advanced Study Institute on Approximation Theory and Spline Functions was held at Memorial University of Newfoundland during August 22-September 2, 1983. This volume consists of the Proceedings of that Institute.

These Proceedings include the main invited talks and contributed papers given during the Institute. The aim of these lectures was to bring together Mathematicians, Physicists and Engineers working in the field. The lectures covered a wide range including Multivariate Approximation, Spline Functions, Rational Approximation, Applications of Elliptic Integrals and Functions in the Theory of Approximation, and Padé Approximation.

We express our sincere thanks to Professors E. W. Cheney, J. Meinguet, J. M. Phillips and H. Werner, members of the International Advisory Committee. We also extend our thanks to the main speakers and the invited speakers, whose contributions made these Proceedings complete.

The Advanced Study Institute was financed by the NATO Scientific Affairs Division. We express our thanks for the generous support.

We wish to thank members of the Department of Mathematics and Statistics at Memorial University who willingly helped with the planning and organizing of the Institute.

Special thanks go to Mrs. Mary Pike who helped immensely in the planning and organizing of the Institute, and to Miss Rosalind Genge for her careful and excellent typing of the manuscript of these Proceedings.

St. John's, Newfoundland, Canada
April 1984

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PRODUCTS OF POLYNOMIALS

Bernard Beauzamy

In this survey paper, we shall present several results concerning estimates for products of polynomials, in one or in several variables.

The results concerning polynomials in one variable are taken from a joint paper of Per Enflo and the author [1]; the results dealing with polynomials in many variables are due to Per Enflo [2]. Though they were proved earlier, it seemed preferable to put them into the last section, since they are technically more complicated. However, the methods of proofs are of different nature, and there is no interdependence between the case of a single variable and the case of several variables.

In the following pages, only outlines of proofs will be given: we refer the reader to [1] and to [2] for detailed proofs.

What we are looking for is estimates of the following type:

$$\|PQ\| \geq \lambda \|P\| \cdot \|Q\| , \qquad (1)$$

where $\|\cdot\|$ is some norm on the space of polynomials (in one or in many variables), and λ is a constant, depending only on the choice of $\|\cdot\|$, and on the choices of the classes C_1 and C_2, in which we will take P and Q respectively.

Let us first deal with polynomials in one variable. There are many norms which are commonly used. Let us mention some of them.

S. P. Singh et al. (eds.), Approximation Theory and Spline Functions, 1–22.

Put $P(n) = a_o + a_i x + \ldots + a_N x^N$. Then, we define:

$$\|P\|_1 = \int_0^{2\pi} |P(e^{i\theta})| \frac{d\theta}{2\pi} ,$$

$$\|P\|_2 = (\int_0^{2\pi} |P(e^{i\theta})|^2 \frac{d\theta}{2\pi})^{1/2}$$

$$\|P\|_\infty = \max_{0\leq\theta\leq 2\pi} |P(e^{i\theta})| .$$

For these three norms, P is considered as a (continuous) function on the Torus Π (that is, the interval $[0, 2\pi]$, mod 2π). So these norms are just the norms of the spaces $L_1(\Pi, \frac{d\theta}{2\pi})$, $L_2(\Pi, \frac{d\theta}{2\pi})$, $L_\infty(\Pi, \frac{d\theta}{2\pi})$.

Another type of norms is obtained the following way. Put:

$$|P|_1 = \sum_0^N |a_j|$$

$$|P|_2 = (\sum_0^N |a_j|^2)^{1/2}$$

$$|P|_\infty = \max_{0\leq j\leq N} |a_j| .$$

This time, these norms can be viewed as norms on sequences spaces: the first one is the norm in ℓ_1, the second in ℓ_2, the third in ℓ_∞, when P is identified with the sequence $(a_0, a_1, \ldots, a_N)$.

The norm $|P|_1$ is sometimes written $\|P\|_{A(\Pi)}$, that is, the norm in the algebra $A(\Pi)$, of functions with absolutely summable Fourier series.

The norm $|P|_\infty$ is also written $\|P\|_{PM}$, norm in the space of pseudo-measures (distributions on Π, the Fourier coefficients of which are bounded; see J. P. Kahane [4]).

The relations between these six norms are as follows:

$$|P|_\infty \le \|P\|_1 \le \|P\|_2 = |P|_2 \le \|P\|_\infty \le |P|_1 .$$

Let us now come back on the estimate (1). Even if we take the simplest norm, $|P|_1$, we cannot hope to have (1) for all poloynomials, that is, without any restriction on the classes C_1 and C_2. Let us give two examples, of independent interest, which will make this assertion clear:

Example 1. Take $P = 1 - x$, $Q_n = \frac{1}{n+1}(1 + x + \dots + x^n)$. Then $PQ_n = \frac{1}{n+1}(1 - x^{n+1})$. Therefore:

$$|P|_1 = 2, \ |Q_n|_1 = 1, \ |PQ_n|_1 = \frac{2}{n+1} \underset{n\to+\infty}{\longrightarrow} 0 .$$

So no estimate like (1) is possible, involving, uniformly, P and the Q_n's.

Example 2. Let $A(\Pi) = \{f = \sum_{j \in \mathbb{Z}} c_j e^{ij\theta}$, with $\sum |c_j| < +\infty\}$, and put $\|f\|_A = \sum_{j\in\mathbb{Z}} |c_j|$.

Take in $A(\pi)$ two functions f and g with disjoint supports:

$$f = \sum_{j\in\mathbb{Z}} c_j e^{ij\theta}, \qquad g = \sum_{\ell\in\mathbb{Z}} d_\ell e^{i\ell\theta},$$

so the product fg vanishes identically. Now put:

$$P_m = \sum_{-m}^{m} c_j\, e^{ij\theta}, \qquad P_m' = \sum_{|j|>m}' c_j\, e^{ij\theta}$$

$$Q_n = \sum_{-n}^{n} d_\ell\, e^{i\ell\theta}, \qquad Q_n' = \sum_{|\ell|>n} d_\ell\, e^{i\ell\theta},$$

so $(P_m + P_m')(Q_n + Q_n') = 0$, and therefore:

$$P_m Q_n = - P_m' g - P_m Q_n',$$

$$\|P_m Q_n\|_A \le \|P_m'\|_A \|g\|_A + \|P_m\|_A \|Q_n'\|_A$$

(using the algebra property of $A(\Pi)$)

$$\le \|P_m'\|_A \|g\|_A + \|f\|_A \cdot \|Q_n'\|_A,$$

and this last quantity tends to 0 when $m, n \to +\infty$, though

$$\|P_m\|_A \underset{m\to+\infty}{\longrightarrow} \|f\|_A, \quad \|Q_n\|_A \underset{n\to+\infty}{\longrightarrow} \|g\|_A.$$

This example can be used to bind from above the constants which will be given later on. Indeed, let $\rho = (\rho_n)_{n\ge 0}$ an increasing sequence of real numbers, satisfying

$$\rho_n \ge 1, \ \rho_{m+n} \le \rho_m \cdot \rho_n, \quad m, n \in \mathbb{N}.$$

One knows that the Beurling Algebra

$$A_\rho = \{f \in A(\pi),\ f = \sum c_j e^{ij\theta},\ \sum_{j\in\mathbb{Z}} |c_j| \rho_{|j|} < +\infty\}$$

is non quasi-analytic if and only if (see for example Y. Domar [3]):

$$\sum_{n\ge 0} \frac{\log \rho_n}{1+n^2} < +\infty .$$

If $(\rho_n)_{n\ge 0}$ satisfies this condition, we obtain from the previous computation $|P_n Q_n|_1 \le G/f_n$, with

$$G = \|f\|_A \cdot \|g\|_{A_\rho} + \|g\|_A \cdot \|f\|_{A_\rho} .$$

This is the case, for example, with $\rho_n = e^{n^\alpha}$, $\alpha < 1$. So we see that the rate of decrease of PQ is much faster than in example 1: we had $|PQ_n| \sim 2/n$, and we have here $|P_n Q_n| \sim G/e^{n^\alpha}$.

There are some simple cases in which estimate (1) is fulfilled. Indeed, fix two integers $m, n \in \mathbb{N}$, and take the classes:

$$C_1 = \{P,\ d^\circ P \le m\}$$

$$C_2 = \{Q,\ d^\circ Q \le n\}.$$

Then the estimate (1) holds, λ depending on m and n. This is clear by the following compactness argument:

Let S_m be the unit sphere of C_1, equipped with the $|\cdot|_1$ norm, and, similarly let S_n the unit sphere of C_2. The product $P, Q \to P \cdot Q$ is a continuous mapping on the product $S_m \times S_n$, which is compact. So it reaches its minimum, that is, there are two polynomials $P^o \in C_1$, $Q^o \in C_2$, such that $|PQ|_1 \ge |P^o \cdot Q^o|_1$, for all $P \in C_1$, $Q \in C_2$. Put $\lambda = |P^o Q^o|_1$, then $\lambda \ne 0$ (since P^o, Q^o are polynomials of given degree, satisfying $|P^o|_1 = 1$, $|Q^o|_1 = 1$). Since (1) is homogeneous, it is proved.

Of course, this type of argument does not give the exact value of λ. But it applies just as well to the other norms: they are all equivalent on the space of polynomials of given degree.

A precise estimate, in this case, of the constant λ is given by the well-known Gelfond's Theorem (see for example M. Waldschmidt [6]):

Gelfond's Theorem. Let P, Q be polynomials of degree m, n respectively. Then $|PQ|_\infty \ge e^{-(m+n)} |P|_\infty \cdot |Q|_\infty$.

Using the equivalence of the norms, one deduces easily from this theorem estimates using other norms.

We shall look mainly at larger classes than those consisting of polynomials of given degree. Indeed, we shall consider polynomials only having some concentration at low degrees. To define this notion, if $P = a_o + a_1 x + \dots + a_N x^N$, and if $m \in \mathbb{N}$, put:

$$P|^m = a_o + a_1 x + \dots + a_m x^m.$$

We say that P has concentration δ (with $0 < \delta \le 1$) at degrees $\le m$ if

$$\|P|^m\|_2 \ge \delta \|P\|_2, \tag{2}$$

which means that

$$\sum_o^m |a_j|^2 \ge \delta^2 \sum_o^N |a_j|^2.$$

For given δ, m, the set of polynomials satisfying (2), with $\|P\|_2 = 1$, is not compact, so no compactness argument can apply.

We need first estimates for a single polynomial, considered as a function on the Torus Π.

1. Estimates for a Single Polynomial

We let m denote the normalized Haar measure on Π, that is $\frac{d\theta}{2\pi}$. The lemma which follows says that, in any set of prescribed measure, there is a large subset on which $|P(e^{i\theta})|$ is large, depending on its first coefficient a_o:

Lemma 1. For each measurable subset $E \subset \Pi$, with $m(E) = a$, and each $\alpha \ge 1$ we have, for every polynomial P:

$$m\{E \cap \{|P(e^{i\theta})| \ge \left(\frac{|a_o|}{\sqrt{e}}\right)^{\alpha/a} \|P\|_2^{1-\alpha/a}\}\} \ge a(1 - 1/\alpha).$$

Proof. We may assume $\|P\|_2 = 1$. We observe that for every measurable subset $A \subset \Pi$, we have:

$$\int_{CA} \log |P(e^{i\theta})|^2 \frac{d\theta}{2\pi} \le \int_{CA} |P(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

$$\le \int |P(e^{i\theta})|^2 \frac{d\theta}{2\pi} = 1,$$

and so, by the classical Jensen's Inequality (see for example

W. Rudin [5]):

$$\int_A \text{Log } |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq \text{Log } \frac{|a_0|}{\sqrt{e}} .$$

Taking $A = E \cap \{|P| < (\frac{|a_0|}{\sqrt{e}})^{\alpha/a}\}$, we get

$$M(A) \leq a/\alpha,$$

from which the Lemma follows.

It should be noted that the result concerns only the measure of the set $E \geq \{|P| \geq (\frac{|a_0|}{\sqrt{e}})^{\alpha/a} ||P||^{1-\alpha/a}\}$, and not the geometry of this set. For example, take $E = \Pi$ (so $a = 1$), take $\alpha = 2$, and $P = 1 + x^N$, so $||P||_2 = \sqrt{2}$, and $a_0 = 1$. We get that

$$m\{|P| \geq e^{-1}\sqrt{2}^{\,-1}\} \geq \frac{1}{2},$$

which is independent of the degree N. But the set $\{|P| \geq (e\sqrt{2})^{-1}\}$ consists in a number of intervals which dependes on N.

The Lemma 1 has an extension, involving the first m coefficients of the polynomial P.

Lemma 2. Let $m \in \mathbf{N}$, $\delta > 0$, $a > 0$, $\alpha \geq 1$. For every polynomial P satisfing

$$|| P|^m ||_2 \geq \delta \, ||P||_2 , \tag{2}$$

and every measurable subset $E \subset \Pi$, with $m(E) = a$, we have

$$m\{E \cap \{|P| \geq \delta^{\alpha/a} (2e)^{-(m+2)(a/\alpha)^{-(m+1)}} ||P||_2\}\} \geq a(1 - \frac{1}{\alpha}).$$

Outline of the Proof

Assume $|| P|^m ||_2 = 1$. If a_0 is substantial, apply lemma 1.

If it is small, look at

$$P_1 = a_1 x + a_2 x^2 + \dots,$$

If a_1 is substantial, apply lemma 1 to P_1: the estimate obtained for P_1 will apply to P, because a_0, being too small, cannot destroy it. If a_1 is too small, pass to

$$P_2 = a_2 x^2 + a_3 x^3 + \dots$$

and so on. By the assumption (2), this process will stop after a number of cases depending only on m. The reader is referred to [1] for details.

Corollary. Under the same assumptions, we have:

$$\|P\|_\infty \geq \delta(2e)^{-(k+2)} \|P\|_2 .$$

We shall now use these results to obtain information about the products of two polynomials.

2. Products of Polynomials

We first consider the case where both polynomials have some concentration at low degrees, thus obtaining an estimate of the type (1):

Theorem 1. Let $m, n \in \mathbf{N}$, $\delta > 0$, $\delta' > 0$. There is a constant $\lambda(\delta, \delta', m, n)$ such that for every polynomial P, Q satisfying:

$$\||P|^m\|_2 \geq \delta \|P\|_2 , \||Q|^n\|_2 \geq \delta' \|Q\|_2 ,$$

we have

$$\|PQ\|_2 \geq \lambda \|P\|_2 \cdot \|Q\|_2 .$$

Proof. Take for E the Torus Π and $\alpha = 3$ in Lemma 1.2. Then, on a set of measure $\frac{2}{3}$, one has: