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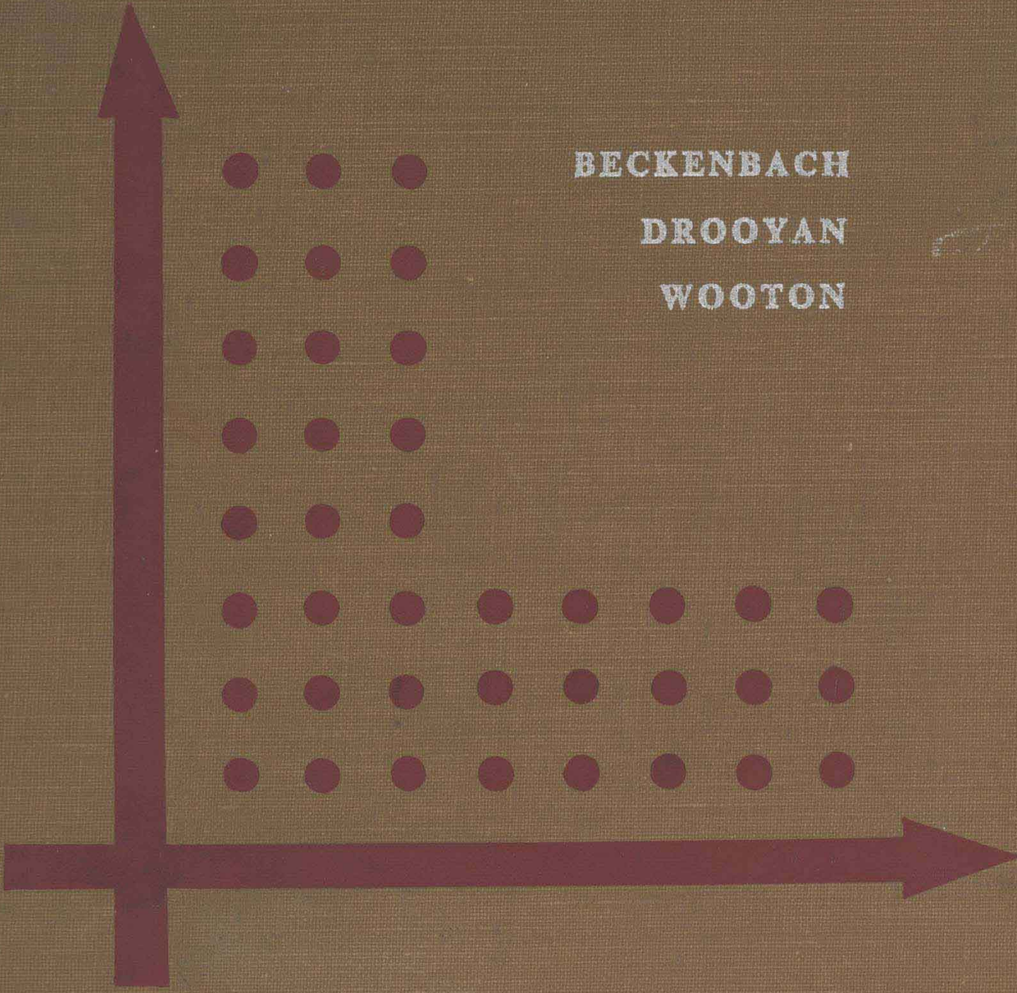

essentials *of* COLLEGE ALGEBRA

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PREFACE

College algebra, as a freshman college course, has come to mean different things in different schools. Courses bearing the name College Algebra, Freshman Mathematics, or something similar, may deal with a wide variety of topics and be aimed at diverse levels of mathematical sophistication. In general, college algebra is not a terminal subject in mathematics, but rather a transitional course, designed to prepare the student for more advanced mathematics in which the algebra will be employed as a tool.

This text is an abridged version of the authors' *College Algebra*, from which a great deal of the material reviewing earlier algebra courses has been eliminated. Thus, although the present text assumes a somewhat stronger prior preparation, it can be taught in fewer semester hours. The material is designed for students who have completed from one and one-half to two years of high school algebra and one year of high school geometry, or equivalent courses offered at the college level, and who propose to continue their study of mathematics.

The content and spirit of the book reflect the recommendations of various curriculum study groups. Structure plays a heavy role in the development, and is used to advantage in introducing matrices, complex numbers, and vectors. Beginning with the set of real numbers in Chapter 1, the student learns that this set comprises a complete field. In Chapter 5, on matrices and determinants, he is introduced to the notions of ring and vector space, and all these structures appear again in Chapter 6 on complex numbers and vectors.

Chapters following Chapter 1 center around the function concept. In particular, determinants are treated as functions of matrices, logarithms are developed from a consideration of the exponential function, sequences are treated through functions having integers as domain, and probability is discussed from a set-function standpoint. Emphasis is given to techniques for readily sketching the graphs of functions.

The traditional topics of plane analytic geometry are covered in Chapters 2, 3, and 4 in sufficient depth that students who have successfully used this text can proceed directly to the study of calculus.

Chapters 4, 5, 8, and 9 are sufficiently independent of each other so that any of these chapters may be omitted for a short course. Furthermore, certain problems, occurring at the ends of various exercise sets throughout the book, ask for proofs omitted in the text; whether or not these problems are assigned will in general determine the level of rigor being established for the course.

Exercises are provided in each section, with the answers to most odd-numbered problems given in the appendix.

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Edwin F. Beckenbach
Irving Drooyan
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1

PROPERTIES OF REAL NUMBERS

1.1 DEFINITIONS AND SYMBOLS

A **set** is simply a collection of some kind. It may be a collection of people, colors, numbers, or anything else. In algebra, we are interested in sets of numbers of various kinds and in their relations to sets of points or lines in a plane or in space. Any one of the collection of things in a set is called a **member** or **element** of the set, and is said to be **contained in** or **included in** (or, sometimes, just **in**) the set. For example, the counting numbers 1, 2, 3, $\cdots$ (where the dots indicate that the sequence continues indefinitely) are the elements of the set we call the set of **natural numbers.**

Sets are usually designated by means of capital letters, A, B, C, etc. They are identified by means of braces in which the members are either listed or described. For example, the elements might be listed as in $\{1, 2, 3\}$, or described as in {first three natural numbers}. The expression "$\{1, 2, 3\}$" is read "the set whose elements are one, two, and three"; "{first three natural numbers}" is read "the set whose elements are the first three natural numbers."

Using the undefined notion of set membership, we can be more specific about some other terms we shall be using.

DEFINITION 1.1 *Two sets A and B are* ***equal****, $A = B$, if and only if they have the same members—that is, if and only if every member of each is a member of the other.*

Thus, if A denotes $\{1, 2, 3\}$, B denotes $\{3, 2, 1\}$, C denotes $\{2, 3, 4\}$, and D denotes {natural numbers between 1 and 5}, then $A = B$ and $C = D$. The phrase "if and only if" used in this definition is simply the mathematician's way of making two statements at once. Definition 1.1 is logically equivalent to the following: "Two sets are equal if they have the same members. Two sets have the same members if they are equal."

DEFINITION 1.2 *If the elements of a set A can be paired with the elements of a set B in such fashion that each element of A is paired with one and only one element of B, and conversely, then such a pairing is called a* ***one-to-one correspondence*** *between A and B.*

For example, if $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, then the sets A and B can be put into one-to-one correspondence in six different ways, two of which are shown:

$$\begin{array}{ccc} \{a, & b, & c\} \\ \updownarrow & \updownarrow & \updownarrow \\ \{1, & 2, & 3\} \end{array} \qquad \begin{array}{ccc} \{a, & c, & b\} \\ \updownarrow & \updownarrow & \updownarrow \\ \{2, & 1, & 3\} \end{array}$$

DEFINITION 1.3 *Two sets are* ***equivalent*** *if and only if a one-to-one correspondence exists between them.*

The symbol "$\sim$" is used to denote equivalence. Thus, $A \sim B$ is read "A is equivalent to B." Intuitively, equivalent sets are sets that contain the same number of members. Clearly, if two sets are equal, they are equivalent, but the converse is not necessarily true—equality of sets requires that the members be identical, not merely that the sets be in one-to-one correspondence.

DEFINITION 1.4 *If every member of a set A is a member of a set B, then A is a* ***subset*** *of B. If, in addition, B contains at least one member not in A, then A is a* ***proper subset*** *of B.*

The symbol "$\subseteq$" (read "is a subset of" or "is contained in") will be used to denote the subset relationship, and the symbol "$\subset$" (read "is a proper subset of" or "is properly contained in") will be used for proper subsets. Thus

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$$

and

$$\{1, 2, 3\} \subseteq \{1, 2, 3\}$$

both make valid usage of $\subseteq$, but $\subset$ is valid only for the first of these pairs of sets. That is,

$$\{1, 2, 3\} \subset \{1, 2, 3, 4\}.$$

Of course, by definition, every set is a subset of itself.

If a set is equivalent to the set $\{1, 2, 3, \cdots, n\}$ for some fixed natural number n, then the set is said to be **finite**. The set that contains no elements is called the **empty set** or **null set,** and is denoted by the symbol $\varnothing$ (read "the empty set" or "the null set"); $\varnothing$ is considered to be a subset of every set, and to be a proper subset of every set except itself. A set that is not the null set and is not finite is called an **infinite set.** For example, the set of *all* natural numbers, $\{1, 2, 3, \cdots\}$, is an infinite set.

DEFINITION 1.5 *Two nonempty sets A and B are* ***disjoint*** *if and only if A and B contain no member in common.*

For example, if $A = \{1, 2, 3\}$ and $B = \{5, 6, 7\}$, then A and B are disjoint.

The symbol $\in$ (read "is a member of" or "is an element of") is used to denote membership in a set. Thus,

$$2 \in \{1, 2, 3\}.$$

Note that we write

$$\{2\} \subset \{1, 2, 3\} \quad \text{and} \quad 2 \in \{1, 2, 3\},$$

but not

$$\{2\} \in \{1, 2, 3\} \quad \text{or} \quad 2 \subset \{1, 2, 3\},$$

since $\{2\}$ is a *subset*, whereas 2 is an *element*, of $\{1, 2, 3\}$.

When discussing an individual but unspecified element of a set containing more than one member, we usually denote the element by a lower-case italic letter (for example, a, d, s, x), or sometimes by a letter from the Greek alphabet: α (alpha), β (beta), γ (gamma), and so on. When symbols are used in this way, they are called "variables."

DEFINITION 1.6 *A* ***variable*** *is a symbol representing an unspecified element of a given set containing more than one element.*

If the given set, called the **replacement set** of the variable, is a set of numbers, then the variable represents a number. Thus

$$x \in A$$

means that the variable x is an (unspecified) element of the set A.

A symbol used to denote the member of a set containing only one member is called a **constant.**

When discussing sets, it is often helpful to have in mind some general set from which the elements of all of the sets under consideration are drawn. For example, if we wish to talk about sets of college students, we may want to consider all college students in this country, or all students in general; or, taking a larger view, we may want to consider students as a special kind of human being—say, all those human beings who are consciously striving to increase their knowledge. Thus, we can draw sets of college students from any one of a number of different general sets. Such a general set is called the **universe of discourse** or the **universal set**, and we shall usually denote it by the capital letter U. It follows that any set in a particular discussion is a subset of U for that discussion.

The slant bar, /, drawn through certain symbols for relations, is used to indicate negation. Thus $\neq$ is read "is not equal to," $\not\subseteq$ is read "is not a subset of," and $\notin$ is read "is not an element of." For example,

$$\{1, 2\} \neq \{1, 2, 3\}, \quad \{1, 2, 3\} \not\subset \{1, 2\}, \text{ and } 3 \notin \{1, 2\}.$$

Another symbolism useful in discussing sets is illustrated by $\{x \mid x \in A$ and $x \notin B\}$ (read "the set of all x such that x is a member of A and is not a member of B"). This symbolism, called **set-builder notation**, will be used extensively in this book. What it does is to name a variable (in this case, x) and, at the same time, to state a condition on the variable (in this case, that x is contained in the set A and is not contained in the set B).

EXERCISE 1.1

Designate each of the following sets by using braces and listing the members.

Example. {natural numbers between 8 and 12}

Solution. {9, 10, 11}

1. {natural numbers between 3 and 10}
2. {natural numbers between 20 and 27}
3. {days in the week}
4. {months in the year}
5. {digits in your age in years}
6. {digits in your home address}

Designate each of the following sets by using set-builder notation.

Example. {even natural numbers}

Solution. $\{x \mid x = 2n, n$ a natural number$\}$

7. {odd natural numbers}
8. {rational numbers}
9. {solutions of $c^x = 5$}
10. {solutions of $x^x = 5$}
11. {elements that are not in the set A}
12. {elements that are in the set B}

Let $a \in A$, but otherwise let a be unspecified. In each of the following, state whether a is a constant.

13. $A = \{6, 7, 8, 9\}$
14. $A = \{6\}$
15. $A = \{\text{natural numbers}\}$
16. $A = \{\text{natural numbers between 2 and 4}\}$

In Problems 17–20, replace the comma between set symbols with either $=$ or $\neq$.

17. {natural numbers less than 3}, {1, 2}
18. {4}, {7}
19. Ø, {0}
20. {5, 7, 9}, {7, 9, 5}

In Problems 21–24, replace the comma between set symbols with either $\in$ or $\notin$.

21. 3, {2, 3, 4}
22. 5, $\{x \mid x \text{ is an even natural number}\}$
23. {2}, {2, 3, 4}
24. Ø, {2, 3, 4}

In Problems 25–28, replace the comma between set symbols with either $\subset$ or $\not\subset$.

25. 5, {4, 5, 6}
26. {5}, {4, 5, 6}
27. Ø, {4, 5, 6}
28. {3, 4}, {4, 5, 6}

29. Let $U = \{5, 6, 7\}$. List the subsets of U that contain
   a. three members
   b. two members
   c. one member
   d. no members

30. Let $U = \{1, 2, 3, 4\}$. List the subsets of U that contain
   a. four members
   b. three members
   c. two members
   d. one member
   e. no members
   f. 2 and one other member

31. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{4, 5, 6, 7\}$, and $C = \{6, 7\}$. Replace the comma in each of the following with either $\subset$ or $\not\subset$.
   a. A, U
   b. C, A
   c. A, B
   d. C, B

32. Let $U = \{\text{natural numbers}\}$, $A = \{\text{even natural numbers}\}$, $B = \{\text{odd natural numbers}\}$, $C = \{x \mid x \text{ is between 1 and 10}\}$, and $D = \{x \mid x \text{ is less than 9}\}$. Which of the following are true?
   a. $A \sim D$
   b. $C = D$
   c. $C \subset D$
   d. $A \sim B$
   e. $A = B$
   f. $D \subset C$
   g. $A \sim U$
   h. A and B are disjoint
   i. $A \subset U$
   j. $C \sim D$
   k. $C \subset A$
   l. C and B are disjoint

33. Let $A \subseteq U$, $B \subseteq U$, $A \subseteq B$, $x \in A$, and $y \in B$.
   a. Is $x \in B$?
   b. Can $y \in A$?
   c. Can there be a $z \in U$ such that $z \notin B$ and $z \notin A$?
   d. Must there be a $z \in U$ such that $z \notin B$ and $z \notin A$?
   e. Must there be a $y \in B$ such that $y \notin A$?

34. Let $A \subset B$, $B \subset C$, $x \in A$, $y \in B$, and $z \in C$.
    a. Can there be a z such that $z \notin B$?
    b. Must there be a z such that $z \notin B$?
    c. Can there be a z such that $z \notin A$?
    d. Must there be a z such that $z \notin A$?
    e. Can there be an x such that $x \notin C$?
    f. Can $A = C$?

35. Consider the results of Problems 29 and 30 in this exercise. Sets containing three elements evidently have 8 possible subsets, and those containing four elements have 16. Determine the number of possible subsets of sets containing one and two elements, respectively, and make a conjecture about the number of possible subsets of a set containing n elements.

36. Explain why the following statement would constitute a valid definition for equality of sets.
    "If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$."

1.2 OPERATIONS ON SETS

Ideas involving universal sets, subsets thereof, and certain operations on sets can be depicted by means of plane geometric figures called **Venn diagrams.** Figure 1.1 is such a diagram, representing a universe having as its

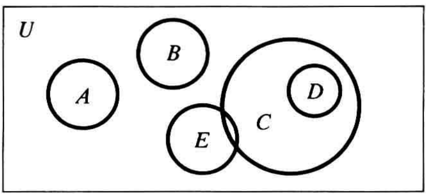

Figure 1.1

elements all points of the rectangle and its interior, and having a number of subsets of the universe denoted by circles and their interiors. In this figure, sets A, B, and C are disjoint, D is a subset of C, and E is neither a subset of C nor disjoint from C.

There are several mathematically important operations on the subsets of a given universe U. A **binary operation** on the subsets of U is a rule that assigns to each pair A and B of subsets of U, taken in a definite order (A first and B second), a third subset C of U. One such operation is defined as follows.

DEFINITION 1.7 *The* **union** *of two subsets A and B of U is the set of all elements of U that belong either to A or to B or to both.*

The symbol ∪ is used to denote the union of sets. Thus $A \cup B$ (read "the union of A and B," or sometimes, "A cup B") is the set of all elements that are in either A or B or both.

Example. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 4, 5, 6\}$, what is $A \cup B$?

Solution. $A \cup B$ is the set of all numbers included in either or both of A and B. Hence,

$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

Notice that each element in $A \cup B$ is listed only once in this example, since repetition would be redundant. Figure 1.2 is a Venn diagram in which the shaded region depicts $A \cup B$ in the preceding example.

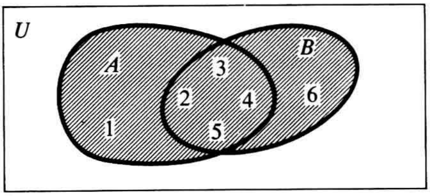

Figure 1.2

A second binary set operation of interest is defined as follows.

DEFINITION 1.8 *The **intersection** of two subsets A and B of U is the set of all elements of U that belong to both A and B.*

The symbol ∩ is used to denote intersection. Thus $A \cap B$ (read "the intersection of A and B," or sometimes, "A cap B") denotes the set of all elements of U that are in both A and B.

Example. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 4, 5, 6\}$, what is $A \cap B$?

Solution. $A \cap B$ consists of those elements that are in both A and B. Hence,

$$A \cap B = \{2, 3, 4, 5\}.$$

Figure 1.3 is a Venn diagram in which the shaded region depicts $A \cap B$ in this example.

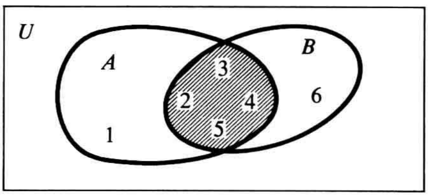

Figure 1.3

The doubly shaded region in Figure 1.4 illustrates

$$(A \cap B) \cap C = \{4, 5\},$$

with $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7\}$.

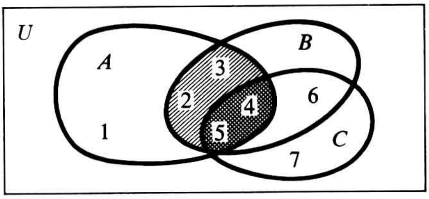

Figure 1.4

Both operations, union and intersection, are binary operations, because they are applied to two sets in relation to each other. The following operation on sets, however, applies to only one set in relation to U.

DEFINITION 1.9 *The* ***complement*** *of a set A in U is the set of all elements of U that do not belong to A.*

The symbol A' (or, sometimes, $\overline{A}$, $\sim A$, or $\tilde{A}$) denotes the complement of A in U.

Example. If $U = \{1, 2, 3, 4, 5\}$ and $A = \{2, 4\}$, what is A'?

Solution. A' contains all members of U that are not in A. Hence,

$$A' = \{1, 3, 5\}.$$

The shaded portion of the Venn diagram in Figure 1.5 represents A' in this example.

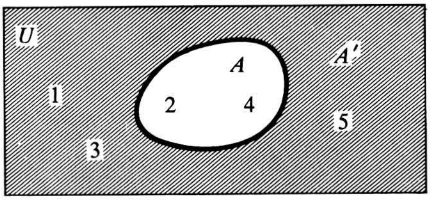

Figure 1.5

EXERCISE 1.2

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5\}$, and $C = \{1, 3, 5, 7, 9\}$. List the members of each of the following.

1. A'
2. B'
3. C'
4. $A \cap B$
5. $A \cup B$
6. $A \cup C$
7. $A \cap C$
8. $A' \cap B'$
9. $A' \cup C'$
10. $(A \cap B)'$
11. $A' \cup C$
12. $C' \cap B$