Peyresq Lectures on

Nonlinear Phenomena

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Preface

Nonlinear science is a very broad domain, with its feet in mathematics, physics, chemistry, biology, medicine as well as in less exact sciences such as economics and sociology. Nineteenth century science was mostly *linear* and the progress towards an understanding of the diverse behaviour of nonlinear systems is among the most important general scientific advances of the twentieth century.

The lectures contained in this book took place at two summer schools, the INLN Summer Schools on Nonlinear Phenomena, in June 1998 and June 1999. The Institut Non Linéaire de Nice (INLN) is a pluridisciplinary institute interested in many aspects of nonlinear science, and the principal purpose of this ongoing series of summer schools is to introduce doctoral students, either from the INLN or form other institutions, to a range of topics that are outside of their own domain of research. The eight courses represented by these lecture notes therefore cover a broad area, describing analytic, geometric and experimental approaches to subjects as diverse as wound-healing, turbulence, elasticity, classical mechanics, semi-classical quantum theory, water waves and trapping atoms. It is hoped that the publication of these notes will be useful to others in the field(s) of nonlinear science.

We would like to take this opportunity, as organizers of the two summer schools, to thank the *Fondation Nicolas-Claude Fabri de Peiresc*, which hosts our stay in the beautiful village of Peyresq in the French Alps, and in particular the president Mady Smets, for providing a wonderfully relaxed atmosphere, allowing the participants and lecturers to interact easily both on scientific and personal levels.

We would also like to thank the local *Direction Régionale du CNRS*, for partially funding these summer schools.

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Valbonne, 2000

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ELASTICITY AND GEOMETRY

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We outline the general principles of thin plate elasticity, by emphasizing their connection with the classical results of differential geometry. The relevant FvK equation, can be solved in some specific cases, even though they are strongly and definitely nonlinear. We present two types of solutions. The first one concerns the contact of a spherical shell on a flat plane at increasing pressing forces, the second one is about the buckling of a thin film under pressure on a flat substrate, where we explain the observed "telephone-cord" pattern of delamination.

1 Introduction

This paper follows from a set of lectures by the two authors given in the beautiful setting of the Rencontres de Peyresq, in the high country, north of Nice in late Spring 1999. Those lectures were devoted to the exposition of some recent results in thin plate elasticity. This venerable field of classical mechanics is witnessing a renewal of interest because in parts of the new attraction of physicists and applied mathematician for everything linking classical geometry and observations made in everyday life. Those lectures were focused first on the general principles of thin plate elasticity, emphasizing as much as we have could their connection with the beautiful results of classical differential geometry. Below, we present a rather detailed derivation of Gauss Theorema egregium, stating the condition under which two surfaces can be mapped on each other without changing the curvilinear distances. Although this is often presented as obvious, the connection between this Theorema egregium and the laws of elasticity of thin plates is not so simple. Hopefully, we make this clearer in our derivation of the equations of Föppl–von Karman for thin plates. Later, we use those equations to analyze two physical problems. The first one concerns the way a spherical shell deforms when pressed on a plane, as when a tennis ball bounces on a racket. We show that two regimes can be observed, depending on the strength of the force. At low forces, the ball makes contact on a flat disc. When this force gets bigger, the ball inverts itself on a cap, and the contact is limited now to the circular ridge in between the inverted and the non inverted part. The details of the geometry of the ridge are deduced from an analysis of the elasticity equations. Finally, we discuss, again by using the same elasticity equations, the problem of buckling of a delaminated film. As it occurs quite often, a film coating some bulk material is under compression because of the way it has been deposited. This film may relax the compression by buckling out of the surface of the bulk material. In many instances a very specific pattern for the buckled film is observed, the so-called telephon cord delamination. We show that this may be explained as a result of a secondary bifurcation of a tunnel like structure of delaminated film, the Euler column.

2 Differential geometry of 2D manifolds

The equations for the elasticity of thin plates (FvK equations later) were derived at the beginning of the twentieth century by Foppl and they are notorious for their complex nonlinear structure. Only recently various investigations put in evidence the possibility of getting explicit solutions in various limits that may be put globally under the heading of large deformations. Actually, those solutions rely heavily on the connection between the FvK equations and the underlying geometry. One central question in this geometry of surfaces, closely linked to elasticity problems, is to find the conditions for a given surface to be isometrically deformable. By this, we mean a deformation leaving unchanged the (intrinsic) distances measured along the surface. If one thinks of a piece of paper this intrinsic distance is just the length of a line drawn between two points on the paper. This length remains the same when the paper is rolled in one way or another, but without tearing, whence the name "intrinsic". Although the definition of this intrinsic length is relatively straightforward in the present case, it becomes rapidly far more subtle when higher dimensions spaces are considered, and even for non planar 2D surfaces (like the surface of a sphere for instance). Riemannian geometry is the geometry of surfaces (and their generalization to higher dimensions, the so-called manifolds) such that the distances are invariant, independent on the coordinates chosen on the surface itself. That this is a crucial question in elasticity theory is evident when noticing that elastic energy precisely accounts for the amount of stretching occured by the material under the deformation. This stretching is measured by how much the distances between material points vary. In the present section, we consider the geometrical problem only, and we shall deal in a rather casual way with deep results of differential geometry related to this question of deformation of surfaces. Far more elaborate presentations of this topic (necessary anyway when dealing with manifolds of dimension higher than 2) can be found in [1].

The problem we shall look at is the following one: under what conditions is it possible to find a one-to-one map between a plane and a surface given by a Cartesian equation $z = Z(x, y)$, without changing the lengths along the surface? Later, we shall also examine the existence of an isometric, one-to-one mapping between two given surfaces. We are looking for a local map of a point of coordinate (x, y) in the horizontal plane to the point of coordinates $x' = x + u(x, y)$, $y' = y + v(x, y)$, $z' = Z(x, y)$, that is situated on the surface. The functions $u(x, y)$ and $v(x, y)$ define "practically" the mapping under consideration. The constraint (imposed on u and v) is that the length element along the surface is the same as the length element on the plane, that is that

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 = ds^2 = dx^2 + dy^2.$$

The orientation of the tangent plane of the surface at the origin $(x, y) = (0, 0)$ can be chosen arbitrarily with the help of a rigid-body rotation. We will therefore assume that it is horizontal. Then, the mapping is close to the identity near the origin. The Taylor expansion of $Z(x, y)$ is quadratic in x and y near the origin, and we expect u and v to be small (actually, u and v are generically cubic in x, y near $x = y = 0$). Expanding dx'^2 and dy'^2 at first order in u and v, and at second order in Z (see explanation below), one gets:

$$\begin{aligned} ds'^2 &= d(x + u(x,y))^2 + d(y + v(x,y))^2 + dZ(x,y)^2 \\ &= dx^2(1 + 2\frac{\partial u}{\partial x} + (\frac{\partial Z}{\partial x})^2) + dy^2(1 + 2\frac{\partial v}{\partial y} + (\frac{\partial Z}{\partial y})^2) \\ &\quad +2dxdy(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial Z}{\partial x}\frac{\partial Z}{\partial y}). \end{aligned} \tag{1}$$

Now the condition of invariance of the length element under the mapping becomes the condition that ds'^2 is the same quadratic form as ds^2, which yields three conditions, one for the coefficient of dx^2 to be one, another for the coefficient of dy^2 to be one too, and the last one for the coefficient of the cross term $dxdy$ to vanish:

$$\frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial Z}{\partial x})^2 = 0, \tag{2}$$

$$\frac{\partial v}{\partial y} + \frac{1}{2}(\frac{\partial Z}{\partial y})^2 = 0, \tag{3}$$

and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial Z}{\partial x}\frac{\partial Z}{\partial y} = 0. \tag{4}$$

As we aim at eliminating u and v, there is one more condition than the number of unknown functions (three versus two), and one condition has to be satisfied for the existence of solutions of (2,3,4). This is to be imposed to the function $Z(x, y)$, a data in the problem. It is obtained by deriving the first equation twice with respect to y, the second one twice with respect to x and the last one once with respect to x and once with respect to y. Subtracting now the last result from the sum of the first two, the u's and v's cancel out and there remains an equation for Z only:

$$\frac{\partial^2 Z}{\partial x^2}\frac{\partial^2 Z}{\partial y^2} - \left(\frac{\partial^2 Z}{\partial x \partial y}\right)^2 = 0. \tag{5}$$

This has a simple geometrical interpretation. Let us write $Z(x, y)$ in the coordinate system diagonalizing its Taylor expansion near $x = y = 0$:

$$Z(x, y) \approx \frac{x^2}{2R_1} + \frac{y^2}{2R_2},$$

where $R_{1,2}$ are the so-called principal radii of curvature of the surface at $x = y = 0$. Therefore, the equation (5) amounts to $\frac{1}{R_1 R_2} = 0$, or equivalently to that at least one of the radius of curvature is infinite. Surfaces such that this holds true everywhere are called developable. When this condition is verified, integration of (2,3) $u(x, y) \approx -\frac{x^3}{6R_1^2}$ and $v(x, y) \approx -\frac{y^3}{6R_2^2}$.

It is a straightforward exercise now to get by the same method the existence condition of an isometry for two smooth surfaces of Cartesian equations $z = Z_a(x, y)$ and $z = Z_b(x, y)$. One can take those two surfaces as tangent to the horizontal plane at $x = y = 0$, then redo the same calculation as before, but by imposing that the length on the two surfaces remain the same under two mappings. Those mapping depend on two functions $u_{a,b}(x, y)$ and $v_{a,b}(x, y)$, and are mappings from the plane to surfaces a and b, such that $x'_a = x + u_a(x, y)$, $y'_a = y + v_a(x, y)$ and $z'_a = Z_a(x'_a, y'_a)$ and a similar set with the subscript b instead of a. Now one imposes that the two length elements $(dx'_a)^2 + (dy'_a)^2 + (dZ'_a)^2$ and $(dx'_b)^2 + (dy'_b)^2 + (dZ'_b)^2$, are the same quadratic form in dx and dy, which yields:

$$\frac{\partial u_a}{\partial x} + \frac{1}{2}(\frac{\partial Z_a}{\partial x})^2 = \frac{\partial u_b}{\partial x} + \frac{1}{2}(\frac{\partial Z_b}{\partial x})^2,$$

$$\frac{\partial v_a}{\partial y} + \frac{1}{2}(\frac{\partial Z_a}{\partial y})^2 = \frac{\partial v_b}{\partial y} + \frac{1}{2}(\frac{\partial Z_b}{\partial y})^2,$$

$$\text{and} \quad \frac{\partial u_a}{\partial y} + \frac{\partial v_a}{\partial x} + \frac{\partial Z_a}{\partial x}\frac{\partial Z_a}{\partial y} = \frac{\partial u_b}{\partial y} + \frac{\partial v_b}{\partial x} + \frac{\partial Z_b}{\partial x}\frac{\partial Z_b}{\partial y}.$$

Because of the obvious similarity of these equations with the one of the previous case, one may use the same method to get rid of the functions $u_{a,b}$ and $v_{a,b}$. One gets at the end that the Gaussian curvatures of the surfaces a and

b have to be the same:

$$\frac{\partial^2 Z_a}{\partial x^2}\frac{\partial^2 Z_a}{\partial y^2} - \left(\frac{\partial^2 Z_a}{\partial x \partial y}\right)^2 = \frac{\partial^2 Z_b}{\partial x^2}\frac{\partial^2 Z_b}{\partial y^2} - \left(\frac{\partial^2 Z_b}{\partial x \partial y}\right)^2. \tag{6}$$

This is the so-called Theorema egregium of Gauss (meaning approximately "outstanding", or "out of the crowd" theorem). This theorem is sometimes said as showing that the Gaussian curvature is a bending invariant: suppose that one can deform the surface isometrically ("bend it"), then its Gaussian curvature must remain the same at every point. This property is obviously a *constraint* on isometric deformations, but it is still in general a difficult question to know if non trivial isometries exist for a given surface. For instance, a plane or a cylinder are deformable surfaces, but not a sphere, nor even a convex surface (when the edge is attached).

In the coming two subsections, we shall expose two questions of differential geometry, the first one having to do with some properties of the developable surfaces, something that will be useful later on for the elasticity of thin plates, the next one will have a more mathematical bent and aims at showing an example of application of the ideas of differential geometry in a well defined case, the so-called Poincaré half-plane.

2.1 Developable surfaces

By definition, such a surface may be mapped on a plane without stretching, and it is C^2 smooth (an important assumption). Let us state first the Theorema egregium in its general form (actually we stated it in the case of almost horizontal surfaces). Its extension is almost trivial, because it only requires to write the Gaussian curvature in an arbitrary system of coordinates. This can be done in a number of ways, and the final result is that the product of the principal curvature (or inverse of the principal radius of curvature) is equal to

$$G(x,y) = \frac{1}{R_1 R_2} = \frac{\frac{\partial^2 Z}{\partial x^2}\frac{\partial^2 Z}{\partial y^2} - (\frac{\partial^2 Z}{\partial x \partial y})^2}{1 + (\frac{\partial Z}{\partial x})^2 + (\frac{\partial Z}{\partial y})^2}, \tag{7}$$

which reduces to the left hand side of (5) when the tangent plane is horizontal.

Therefore the algebraic condition that the Gaussian curvature vanishes is always

$$\frac{\partial^2 Z}{\partial x^2}\frac{\partial^2 Z}{\partial y^2} - \left(\frac{\partial^2 Z}{\partial x \partial y}\right)^2 = 0.$$

The algebra will be made simpler later on with the notation

$$[Z_a, Z_b] = \frac{\partial^2 Z_a}{\partial x^2}\frac{\partial^2 Z_b}{\partial y^2} + \frac{\partial^2 Z_b}{\partial x^2}\frac{\partial^2 Z_a}{\partial y^2} - 2\frac{\partial^2 Z_a}{\partial x \partial y}\frac{\partial^2 Z_b}{\partial x \partial y},$$

so that

$$G(x,y) = \frac{[Z,Z]}{2(1+(\frac{\partial Z}{\partial x})^2 + (\frac{\partial Z}{\partial y})^2)}.$$

A classical problem, called the Monge-Ampère equation amounts to find the unknown function(s) [a] $Z(x,y)$ such that $G(x,y)$ is prescribed in a bounded domain for instance. In the case of zero curvature, the Monge-Ampère equation $G(x,y) = 0$ has an interesting general solution. Consider a one parameter family of planes of Cartesian equation:

$$\zeta(x, y|s) = a(s) + b(s)x + c(s)y,$$

where $a(s)$, $b(s)$ and $c(s)$ are smooth arbitrary functions of a parameter s. The envelop of this family of planes is the surface tangent everywhere to one plane in the family. As a first result, we show that this surface is tangent to those planes along straight lines (the generatrices). Consider two planes with neighboring indices, s and $s + \delta s$, δs small. Those two planes cross along a straight line, intersection of planes of equation

$$\begin{aligned} & z = a(s) + b(s)x + c(s)y, \\ \text{and} \qquad & z = a(s+\delta s) + b(s+\delta s)x + c(s+\delta s)y. \end{aligned}$$

Take the difference between those two equations and divide by δs, then one obtains that, in the limit of a vanishing δs, the limit line of intersection of the two planes has Cartesian equation:

$$z = a(s) + b(s)x + c(s)y, \tag{8}$$

$$\text{and} \qquad \frac{da}{ds} + \frac{db}{ds}x + \frac{dc}{ds}y = 0. \tag{9}$$

These are two Cartesian equations of planes, showing that the envelope of the family of planes must include generically the straight lines whose equation is obtained in this way. It remains to show that the surface so generated has zero Gaussian curvature. This is shown directly, by computing $G(x,y)$ as given by (7). The calculation is not straightforward, and so we shall decompose

[a] There might be more than one solution: think of the case of zero Gaussian curvature.