LINEAR ALGEBRA

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preface

This book is, in general, organized to follow closely the CUPM recommendations for a course in linear algebra. A discussion of "exterior product" is an added feature.

While it is presumed that students using this book will have had analytic geometry, they need not necessarily have used the vector approach. A considerable amount of material is therefore devoted to the vector analytic geometry of lines and planes in 3-space. A complete discussion of two- and three-dimensional vector algebra is included as background material. This lower-dimensional work is generalized to an n-dimensional treatment.

Determinant theory is treated axiomatically, with parallel discussion of the exterior product used to define determinants inductively. The exterior product again plays an important role in the solution of systems of linear equations. While the definition of "exterior product" is not the most general one, it is quite adequate for extensive study of linear dependence.

The discussion of vector spaces is devoted primarily to the finite-dimensional case, although there is an introduction to the general case. Orthogonal and orthonormal bases are discussed, including the Gram-Schmidt method for constructing the latter.

The concepts of matrix similarity, equivalence, and congruence are discussed, and are from the outset related to the appropriate linear or bilinear function. Eigenvalues and vectors are introduced and are used in the reduction of conics and quadric surfaces to canonical form. Canonical forms, including the classical and Jordan forms, are derived for each of the above matrix relations.

The definition-theorem proof approach is taken throughout, giving the student insight into the mathematical structure of the subject, as well as a background for later pure and applied courses in which linear algebra plays a role.

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chapter 1
SETS

contents

1.1 *SETS*

In any branch of knowledge a minimum number of undefined concepts is sought in terms of which all other concepts may be defined. The set concept in mathematics is one of these. The word "set" means a "collection," "group," or "aggregate." These words are synonyms rather than definitions, since the set concept is so basic that it must be accepted as "undefined" or "intuitive." It can, however, be illustrated—for example, the set of men who have been governors of New York; the set of all elephants in the St. Louis zoo; the set of all satellites orbiting around the Earth; the set of all real numbers; the set of all triangles with area one square inch; the set of objects on exhibit at the Chicago Art Museum at 10:00 P.M., September 5, 1965. The elements or objects that make up a set are called its *members*, and we say that *each member belongs to the set.* This concept of membership in a set must also be undefined. The important characteristic of any set is that each object in the universe either is, or is not, a member of a given set. The set consists of those objects and only those that are members of the set. A set may have no members—for example, the set of all women Presidents of the United States. Each person chosen satisfies the second alternative with respect to being a member of this set; that is, each person (or object) either must be (or have been) a woman President of the U.S. or is not (or has not been) a woman President of the U.S. In each case the second alternative is the correct one. Such a set with no members is called the *empty* or *null* set.

1.2 *CONSTANTS AND VARIABLES*

A *constant* is the name for a specific member of a set. For example, "Hubert Humphrey" is a constant because it is the name of a specific member of the set "the Vice-Presidents of the United States." The symbols "2" and "4" are constants because they are names for specific members of the set of "even integers." There may be other constants that are names for the same members of the set; thus "1 + 1" and "1 + 1 + 2" are also constants that are names for the numbers represented by 2 and 4, respectively.

A *variable* is a symbol that is a name of one or more members of a set; thus a variable can be replaced by one or more constants. The set associated with the variable is called the *range of the variable.* The range is not always explicitly stated, but it can be inferred. For example, ______ was a Vice-President of the United States. The range probably is the set of Vice-Presidents of the United States living or dead and the symbol "______" can be replaced by any one of a number of constants or names, such as Richard Nixon, Alben Barkley, or Henry Wallace. If one is not concerned with the veracity of the resulting sentence, then the range might be the set of all people living or dead. This would allow the replacement "Nelson Rockefeller" for "______," which would result in an untrue statement.

Consider the following example: "The set of all x such that $(x + 1)^2 = x^2 + 2x + 1$." The variable is x and the range could be the set of real numbers. The statement means that the variable x can be replaced by any constant, such as 2, $\sqrt{3}$, or π, etc., resulting in different names for the same real number for both members of the equality. Note that the symbol "=" means that both members are names for the same number.

1.3 *SET BUILDERS: EQUATIONS AND INEQUALITIES*

One method of defining sets is by means of equations. For example, solving the equation $x^2 + x - 2 = 0$ is equivalent to finding the set of all real numbers x such that $x^2 + x - 2 = 0$. This is denoted symbolically by $\{x \mid x^2 + x - 2 = 0\}$. In general, the notation $\{x \mid \text{condition}\}$ means the set of all x that satisfy the condition written after the vertical line. Solving this quadratic by the usual methods shows that 1 and -2 constitute the solution set. That is, if x is replaced by either 1 or -2, then $x^2 + x - 2$ and 0 are different names for the same number, and if x is replaced by any other number, they are names of different numbers. In other words, the set consisting of 1 and -2 and the set $\{x \mid x^2 + x - 2 = 0\}$ are identical sets. The notation $\{1, -2\}$ is used to denote the set consisting of the members inside the braces. We say that two sets are *equal* if they consist of exactly the same elements. Thus the above remarks can be written symbolically in the form: $\{x \mid x^2 + x - 2 = 0\} = \{1, -2\}$.

If the range of x is the set of real numbers, consider the set $\{x \mid x^2 + 1 = 0\}$. Since the square of any real number is at least 0, there are no such real x's. The set is, therefore, the *null* set. If the null set is denoted by $\emptyset$, this can be written symbolically in the form: $\{x \mid x^2 + 1 = 0\} = \emptyset$.

Inequalities can also be used to build sets. For example, if the range of x is the set of integers, consider the set $\{x \mid x^2 + x - 2 < 0\}$. The members of this set can be found by factoring $x^2 + x - 2 = (x + 2)(x - 1)$. In order for the product of two numbers to be negative, one factor must be negative and the other positive. By making a few trials, one soon sees that if x is replaced by -1 or 0, this will be the case and that it will not be the case for any other replacements. Thus $\{x \mid x^2 + x - 2 < 0\} = \{0, -1\}$. A systematic way of solving inequalities will be discussed later.

EXERCISE 1.3

1. Which of the following are names of positive one-digit integers? 3, $\sqrt{3}$, 3^2, $\sqrt{4}$, 4^2, $\sqrt{16}$, 64^3, $25^{3/2}$, $9 + 3$, $8 \div 3$, $125 \div 25$.
2. Can a set have an infinite number of members?
3. In the statement, "______ are integers whose sum is 4," what would be the inferred range of the variable?
4. Give an example of a set with the following number of members: 1, 3, 7.
5. Give a numerical example of one sentence in which two variables occur.
6. If $S = \{1, 2, 3, 4, 5\}$, give an example of a constant.
7. Does the following example illustrate equal sets:
$$\{x \mid x^2 + 3x - 4 = 0\} = \{-4\}?$$
8. What is the set of even prime integers?
9. What is the set of one-digit prime numbers?
10. What range is inferred by the following set: "______ was one of the original thirteen colonies."
11. Name one constant in the set of Presidents of the United States.
12. Does $\emptyset = \{0\}$?
13. Give an example of three constants that are all names of the same member of a set.
14. Write in words a sentence that is expressed symbolically by
$$\{x \mid x^2 - 6x - 7 < 0\} = \{0, 1, 2, 3, 4, 5, 6\}.$$
15. What are the elements of the set $\{x \mid x^2 - 2x + 1 = 0\}$?
16. Consider the symbolic statement $\{x \mid x^3 - 1 = 0\} = \{1\}$. Give a range that makes this a true statement and another range that makes it false.

1.4 *RELATED SETS*

Two sets may be related in some manner. For example, if A is the set of Presidents of the United States and B is the set of states of the United States, then the relationship between the Presidents and the states where they were born can be indicated by pairings as follows: (Washington, Virginia), (John Adams, Massachusetts), (Jefferson, Virginia), etc. In each pairing the first element is the name of the President and the second is the name of the state where he was born. These pairings are called *ordered pairs*, since the President is always written before the state. A set of ordered pairs is called a *relation*. Another example of a relation is the 1962 football schedule of the Western Big Five conference. In this case both sets, A and B, are the set of teams in the Big Five conference: Washington, California, Stanford, U.C.L.A., and U.S.C., and the relationship is that the team in set A plays the team in set B as a home game for the team in set A. The relationship is: (California, U.C.L.A.), (California, Stanford), (U.S.C., California), (U.S.C., Washington), (Stanford, Washington), (Stanford, U.S.C.), (U.C.L.A., Stanford), (U.C.L.A., U.S.C.), (Washington, California), (Washington, U.C.L.A.). For example, U.S.C. played California and Washington at home, as indicated by the set of ordered pairs above.

If a relationship has the property that no member of a set A occurs in two ordered pairs, i.e., no two ordered pairs have the same first member, then the relationship is called a *function*. In the two examples above, the first relationship is a function because no name of a President occurs more than once as the first member of an ordered pair of the relation. The relation representing the football schedule is not, however, a function since, for example, California occurs as the first member of two ordered pairs. In fact, in this example each name of a school occurs twice as a first member. In a function, the name of a member of the second set may occur more than once. Thus Virginia occurs twice in the first three pairs.

In any function, the variable in first position is called the *independent variable*, and the variable in second position is called the *dependent variable*. Thus in the example above, "name of a President of the United States" is the independent variable, and "name of the state where he was born" is the dependent variable. The sets A and B from which the first and second elements are chosen are the ranges of the independent and dependent variable, respectively. The set of first elements of the function is called the *domain of the function* and the set of second elements is called the *range of the function*.

It is important to realize that a relation is defined by the set of ordered pairs and that no explanation need be given as to how they were constructed. For example, the set $\{(1, 2), (3, 7), (-2, 3), (1, 5), (-6, 7)\}$ is a relation in which the ranges of the first and second variable are both integers. Note that this relation is not a function since 1 occurs twice as a first member. Simple functions can be generated by indicating a sequence of fundamental operations to be performed. For example, $(x + 1)^2$ from the preceding section indicates an addition followed by a multiplication. The function generated is the set of ordered pairs, $\{(x, (x + 1)^2)\}$, where the first member can be any real number and the second member is the number obtained by squaring the result of adding one to the first member. For example, the ordered pair of the function with 3 in first position is $(3, (3 + 1)^2)$ or (3, 16). For short, we say "the function $(x + 1)^2$," rather than "the function $y = (x + 1)^2$," in which the variable y is simply another name for the second member of the ordered pair.

If a function has the property that no element of the range occurs twice, then the function is said to *establish a 1–1 correspondence* between the domain and range of the function. For example, the function {(1, 11), (2, 12), (3, 13), (4, 14), (5, 15)} establishes *a 1–1* correspondence between the sets {1, 2, 3, 4, 5} and {11, 12, 13, 14, 15}. If the set of positive integers of natural numbers is the domain of the function generated by $2n$, then this function establishes a 1–1 correspondence between the natural numbers and the even natural numbers. A few of the pairs of this function are (1, 2), (2, 4), (3, 6), ... It establishes a 1–1 correspondence, because no even integer can occur as a second member in two different pairs, since for this to happen two different natural numbers would have to become equal after doubling.

1.5 *THE NUMBER LINE*

A fundamental relationship between algebra and geometry is established as follows. Let L be any straight line, and let there be a function whose domain is the set of real numbers and whose range is the set of points on the line L. Let the function be denoted by $\{(p, P)\}$, where p is a real number, and P is the point on the line L corresponding to p in the function. The fundamental axiom assumes that there exists such a function that establishes a 1–1 correspondence between the real numbers and the points of L, and which has the following properties. In particular, if $(0, O)$ and $(1, U)$ are two ordered pairs of the function, then O and U are different points. They are called, respectively, the origin and the unit point. Furthermore, it is assumed that the point N of (n, N), where n is a positive integer, lies on the same side of O as U and is n times as far from O as U. Similarly the point N' of $(-n, N')$ occurs on the opposite side of L from U at a distance n times as far from O as U. The point R of $(p/q, R)$, where p and q are positive integers, is located on the same side of O as U and such that P of (p, P) is q times as far from O as R, i.e., the segment (O, P) is divided into q equal parts, and R is one end point of the part whose other end point is O. The point R' of $(-p/q, R')$ is similarly located on the other side of O. Finally, if S, S', and S'' correspond to arbitrary real numbers r, r', and r'', i.e., (r, S), (r', S'), and (r'', S'') are ordered pairs of the function, such that $r' < r < r''$ (r' is less than r which is less than r''), then S is between S' and S'', and the direction from S' to S'' is the same as that from O to U. Thus each point of L corresponds to a real number in such a way that the orderings of the numbers and of the points on the line correspond.

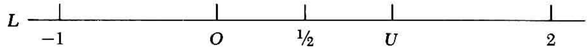

Figure 1.5.1

This correspondence between real numbers and points on the line makes it possible to represent any set of numbers by a picture or graph on the line. This is the first case of the *fundamental principle of analytic geometry*, which states that the graph of a set of numbers is the totality of points on a line that correspond to these numbers. Conversely, any set of points on the line can be described or analyzed algebraically by considering the set of numbers corresponding to the points in the above function. Analytic geometry may be defined as the study of geometry through algebra by making use of the fundamental principle. The principle applied to sets of points in the plane will be discussed later.

1.6 *LOGIC*

Before beginning the study of sets, a basic tool of mathematics, we need to define certain logical connectives in terms of their truth values. These definitions are a part of the study of logic. The fundamental concept of logic is that of a (declarative) *sentence*, which is either true or false but which cannot be both true and false. For example, "This paper is white" is a declarative sentence, but "How are you" is not.

A new sentence can be formed from another by *negation*; that is, by use of the word "not." For example, "17 is not divisible by 4" is obtained from "17 is divisible by 4" by negation. If sentence A is true, its negation is false; if A is false, its negation is true. This can be shown conveniently by means of a *truth table* in which we list each possible truth value of A and the corresponding truth value of its negation.

A	*negation of A*
T	F
F	T

Two sentences may be combined by means of a connective to form a new sentence whose truth value will be defined in terms of the truth values of each of its component sentences. One such connective is *conjunction*, which is usually denoted by "and." For example, "All horses are black and all live horses have hearts." A conjunction of two true sentences is true; if either or both component sentences are false, the conjunction is false.