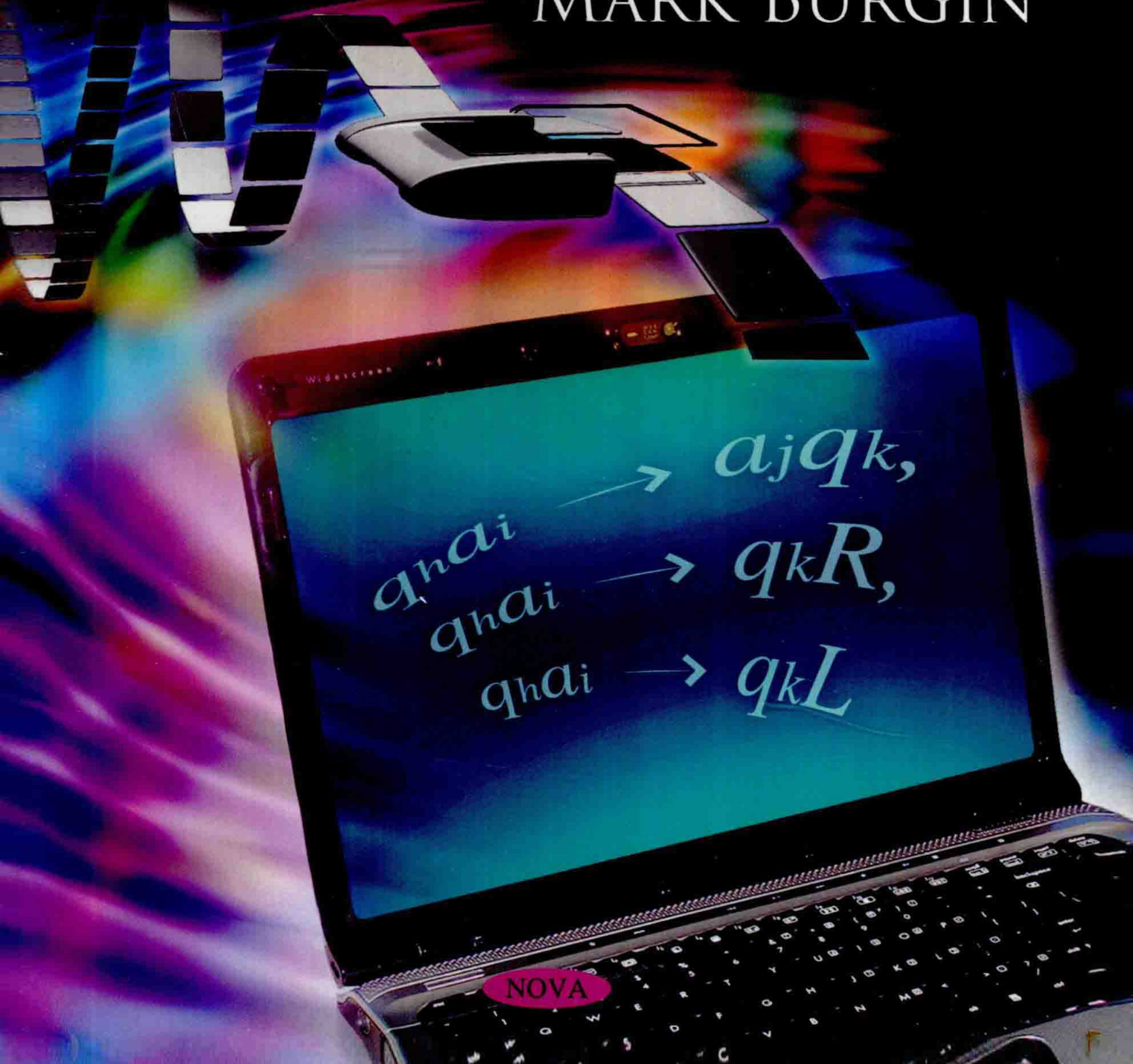
Measuring Power of Algorithms, Computer Programs and Information Automata

Mark Burgin



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ANNOTATION

We live in the world oversaturated with versatile artificial information processing systems, such as computers, networks, embedded devices and hand-held gadgets. Thus, we need to know how all these "smart" machines influence us and our environment, as well as what are our abilities in developing and controlling them. The world of information processing devices is very complex. History of human civilization persuasively demonstrates that to correctly understand and efficiently deal with very complex systems, people need intellectual tools called theories. This book studies two key problems in the framework and by means of the theory of algorithms, automata and computation. The first one is to find what algorithms (represented by computer programs) and automata (representing computers and computer networks) can do. Namely, we are interested in what problems can be solved and what decisions can be made by artificial information processing systems. The second problem is to understand what it is possible to do with algorithms and automata. Possibilities to perform useful operations, such as optimization, minimization, totaling and approximation, with computers, networks, embedded devices, hand-held gadgets and their software are studied. Power of algorithms and automata is treated in a general theoretical context utilizing the multiglobal axiomatic approach. Examples of applications of the obtained theoretical results to software correctness are also presented.

The study of algorithms and automata is conducted not for some specific classes of algorithms and automata, for example, for such as popular classes of Turing machines or partial recursive functions, but it is done in the axiomatic setting where instead of restricting our findings to one class of algorithms or automata, axioms provide a powerful flexibility for theoretical explorations and practical applications. As a result, we compress existing knowledge about algorithms and automata unifying the theory of algorithms, automata and computation because axiomatic results allow one to obtain a quantity of specifications for particular classes of algorithms and automata.

With its more than a hundred theorems, more than hundred and twenty propositions and more than two hundred and fifty corollaries, some of which are theorems of other authors, the book is a profound source of theoretical knowledge in computer science and in the theory of algorithms, automata and computation.

PREFACE

"That's very curious!" she thought.
But everything's curious today."
Lewis Carrol, Alice in the Wonderland

We live in the world where computers, networks and embedded devices underpin a large amount of everyday life, as well as many individual and social endeavors. Computers are prevalent in science, education, business, and industry. They have come to entertainment and politics. Living in the world saturated by computers, networks, and other information processing systems people rely on them more and more. Thus, not to come to unexpected disasters, it is necessary to understand and know what computers and networks are doing, what they can do and what they cannot achieve, which problems are solvable by computers and which are beyond their power, what is possible to accomplish using computers and networks and what is beyond our reach even when we use these wonderful devices, which extend power of people in an unbelievable way.

The world of computers and their applications is very complex and sophisticated. It involves interaction of many issues: social and individual, biological and psychological, technical and organizational, economical and political. Complexity of the world of modern technology is reflected in a study of Gartner Group's TechRepublic unit (Silverman, 2000). According to it, about 40% of all internal IT projects are canceled or unsuccessful, meaning that an average of 10% of a company's IT department each year produces no valuable work. An average canceled project is terminated after 14 weeks, when 52% of the work has already been done, the study shows. In addition, companies spend an average of almost $1 million of their $4.3 million annual budgets on failed projects.

However, humankind in its development created a system of intellectual "devices" for dealing with extremely complex systems. This system is called science and its "devices" are theories. Science is the only efficient tool for dealing with this overwhelming complexity.

When people want to see what they cannot see with their bare eyes, they build and use various magnifying devices. To visualize what is situated very far from them, people use telescopes. To discern very small things, such as microbes or cells of living organisms, people use microscopes. In a similar way, theories are "magnifying devices" for mind. They may be utilized both as microscopes and telescopes. Being, as a rule, very complex, these "theoretical devices" have to be used by experts. That is why IT companies might be able to minimize canceled projects, as well as to reduce time for the necessary cancellation, if they have relevant evaluation theory and consult people who know how to apply this theory.

All advanced theories have a mathematical ground core because one of the most efficient tools to harness complexity is mathematics and theoretical investigation based on mathematics. Thus, mathematics helps science and technology in many ways. Mathematical methods play more and more important role in society, helping people to solve their problems, to obtain new knowledge and to better understand the world people live in. Mathematics is applied to a diversity of practical fields. Scientists are even curious, as wrote the Nobel Prize winner Eugene Wigner (1959), why mathematics being so abstract and remote from reality is unreasonably effective in the natural sciences. It looks like a miracle.

One of the more efficient tools that mathematics is permanently producing is a host of algorithms for solving various problems. Although many people do not see and understand algorithms, they are everywhere. Algorithms wired in computers and embedded devices increasingly pervade people's vehicles, ships, aircrafts, cell phones, TV sets, copying machines, medical devices, various appliances, social organizations, medical facilities, colleges, big and small companies, libraries, and so on and so forth. Solving more and more complex problems, algorithms themselves are also becoming increasingly and seemingly inexorably more complex, demanding advanced theoretical tools for their understanding, development and utilization.

In addition, algorithms are used by people and organizations even without computers and embedded devices. For instance, algorithms have become instruction manuals for a host of routine consumer transactions. When people buy some product, e.g., a book, TV set, clothes or furniture, algorithms tell what is necessary to do. Management and production algorithms are used by companies which produce and/or sell various products. Companies that use better algorithms have higher profit. When the Internet is used for selling and buying, algorithms direct people's actions and control at the same time reactions of the Internet and used computers.

Computers, embedded devices and their networks are directed and regulated by algorithms. Consequently, to know what computers and networks can do, it is necessary to evaluate abilities of algorithms, and we need more powerful methods and techniques because complexity of systems created and studied by people grows beyond all imaginable limits. Computers, their software and their networks are among the most complicated artificial systems of our time and it is not an easy task to understand and properly use their potential.

Mathematicians and computer scientists developed cogent and effectual means for solving this problem, creating the theory of algorithms, automata and computation. However, in this theory different classes of algorithms and automata are studied separately, providing only occasional connections between these classes. As a consequence, many similar results are obtained independently and demand individual proofs of their validity. This book extends tools of this theory, unifying a multiplicity of results for particular classes of algorithms, automata.

One of the most powerful methodologies developed in mathematics and transmitted to science is the axiomatic approach. It has demonstrated its power in many areas of mathematics. We show here that axiomatic methods can be very efficient for the theory of algorithms and computer science when they are applied to a variety of problems, especially, to problems of information technology and computational practice. One of the advantages of the axiomatic approach is that it works not only for computers, embedded devices and software systems, but also for their networks, such as Internet or the Grid.

Here we further develop an axiomatic approach in computer science initiated by Floyd, Hoare, Manna, Blum and other researchers. The axiomatic context allows a researcher to explore not only individual algorithms and separate classes of algorithms and automata but also classes of classes of algorithms, automata, and computational processes. As a result, axiomatic approach goes higher in the hierarchy of computer and network models, reducing in such a way complexity of their study. The suggested axiomatic methodology is applied to evaluation of possibilities of computers, their software and their networks with the main emphasis on such properties as computability, decidability, and acceptability.

The main goal of this book is to demonstrate how axiomatic methods in the framework of projective mathematics work in computer science, efficiently achieving a high abstract level for a comprehensive unification of many important results in the theory of algorithms, automata and computation. At the same time, in spite of the advanced mathematical core, methods and technique developed here are oriented at various practical problems, such as software and hardware verification and testing.

Chapters 1 and 2 and Section 3.1 describe methodology and philosophy of the projective axiomatic theory of algorithms, automata and computation. Mathematical results of this theory are concentrated in Chapters 4 through 8 and Section 3.2. Used concepts and structures are mostly explained in the Appendix. However, understanding of the mathematical material demands sufficient knowledge of basic mathematics, mathematical reasoning and acquaintance with the conventional theory of algorithms, automata and computation.

It is necessary to stress that the development of the projective axiomatic theory of algorithms, automata and computation is at the very beginning, providing a rich area for further research.

The book is a mathematical monograph, which can be used as an addition to the senior undergraduate course on algorithms, automata, formal languages and computation or as the textbook for a separate graduate course.

CONTENTS

Chapter 1

INTRODUCTION

The only way to predict the future is
to have power to shape the future.
Eric Hoffer (1902-1983)

In the first section, we consider philosophical aspects of power structure to put problems of algorithm and automaton power in the general context of power. Then we discuss different research approaches in mathematics and computer science to the development of mathematics and problems in information technology. We also explain what is new in the approach adopted in this book and how it is related to other directions in mathematics and computer science. In the last section, we describe the structure of this book.

1.1. DIMENSIONS OF POWER

The sole advantage of power
Baltasar Gracian y Morales (1601 - 1658)

Power is about being able to do definite things. If we look into a dictionary (cf., for example, (Merriam-Webster Online Dictionary, 2009)), we can see the following definitions of power.

Power is:

- ability to act or produce an effect;
- legal or official authority, capacity, or right;
- possession of control, authority, or influence over others;
- might, physical strength;
- mental or moral efficacy.

In each of these definitions, we find that power is always related to some area and has parameters, which allow one to measure, or at least, to estimate, power. In some cases, these parameters are informal and there are no exact measures for estimating them. For instance, taking power as influence of people on other people and on society, we have only some rather vague indicators of this power.

At the same time, taking the concept of power in physics, we have a very strict definition and measures for power. Namely, power is the rate at which work is done or energy is transferred. It is measured in watts equal to joule per second. In mathematical terms, power is the derivative of work or energy transfer with respect to time.

Saying that power is a capacity to do definite things, we reflect only the most evident aspect of power. A deeper analysis of the concept of power shows that power also reflects an ability not to do definite things. For instance, people who serve in the army do not have power to do many things on themselves – they have to obey orders of their commanders.

We see that this aspect of power, we can call it power of inaction, is associated with the notion of the *free will*, which has been frequently discussed in philosophy and theology. It looks like the *free will* has nothing to do with computers, algorithms, and automata. However, even such a human-related (anthropic) notion as *free will* has its projections on properties of computers, algorithms, programs, and automata. Namely, for algorithms, computer programs, computing automata and information processing systems, such as computers and networks, power is evaluated by what they are able to compute (in general, able to do) and what they cannot compute (in general, cannot do). These questions have been intensely studied based on scientific reasoning and mathematical models (cf., for example, (Turing, 1951; Dreyfus, 1979; Penrose, 1989; 1994; Cleland, 1993; Copeland, 1998; Lewis, 2001; Burgin, 2005)). Here we use a logical relativistic approach to these problems. It allows us to consider and solve them in more generality than standard approaches do.

To better understand the concept of power and its specific features when power is related to algorithms, programs and automata, let us consider the structure of the world and the role of power in this world.

We all live in the physical (material) world and many perceive that this is the only reality that exists. However, in such philosophical and religious systems as Buddhism or Hinduism, physical reality is treated as a great illusion and the only true reality is the spiritual world. According to the reality stratification developed in (Burgin, 1997), it is possible to consider the spiritual world as a part of a higher layer of the mental world. In contrast to the conventional point of view, the mental world has many levels. As contemporary psychology states, each individual has a specific inner world, which is based on the psyche and forms mentality of the individual. These individual inner worlds form the lowest level of the mental world, which complements our physical world. On the next level, we have a group mentality, that is, the mentality of communities and society as a whole. There are also higher levels of the mental world (Burgin, 1997).

Some thinkers, following Descartes, treat the mental world as independent of the physical world. Others assume that mentality is produced by physical systems, such as the brain. In any case, the mental world is different from the physical world and constitutes an important part of our reality. Moreover, according to contemporary physics, our mentality influences the physical world and can change it. We can see, for example, how ideas of people change our planet, create many new things and destroy existing ones. Besides, physicists, who research the very foundation of the physical world, developed the concept of *observer-created reality* interpretation of quantum phenomena (cf., for example, (Herbert, 1987)).

In any case, as the dualistic model is not complete, the third world, the world of ideas (or pure forms), was postulated by Plato (429-347 B.C.E.). In spite of the attractive character of Plato's brainchild, the majority of scientists and philosophers believe that the world of ideas does not exist, because nobody has had any positive evidence in support of it. The crucial argument of physicists is that the main methods of verification in modern science are observations and experiments, and nobody has been able to find pure forms or ideas by means of observations and experiments. In spite of this, some outstanding modern thinkers, such as philosopher Karl Popper (1902-1993), mathematician Kurt Gödel (1906-1978), and physicist Roger Penrose (1931-), continued to believe in the world of ideas, giving different interpretations of this world but suggesting no means for their experimental validation. Here we also adhere to the idea of the triadic world stratification because the development of science and mathematics allows a new understanding and interpretation of the world of ideas, giving efficient tools for its validation and clarification. Before we explain the new concepts and related to them objective phenomena, let us consider classical conceptions.

In the Platonic tradition, the *global structure* of the world has the form of three interconnected worlds: *material world*, *mental world*, and the *world of ideas* or *forms*.

Popper describes the world structure in a different way:

World 1: Physical objects or states.
World 2: Consciousness or psychical states.
World 3: Intellectual contents of books, documents, scientific theories, etc., which is treated as scientific knowledge.

Other authors refer World 3 to signs in the sense of Charles Pierce, although they do not insist that it consists of objects that Pierce would classify as signs (cf., for example, (Skagestad, 1993; Capuro and Hjorland, 2003)).

Only recently, modern science made it possible to achieve a new understanding of Plato ideas, representing the *global world structure* in the form of the existential triad of the world. In this triad, the material world is interpreted as the physical reality, while ideas or forms are associated with structures, and the mental world encompasses much more than individual mentality or conscience (Burgin, 1997; Burgin and Milov, 1999). In particular, the mental world includes social mentality (conscience). In this stratification, the World of structures not only impersonates Plato's world of ideas but also includes Popper's World 3 as knowledge is a kind of structures that are represented in people's mentality (Burgin, 2004).

Thus, the *existential triad* of the world (the world's global structure) has the following form:

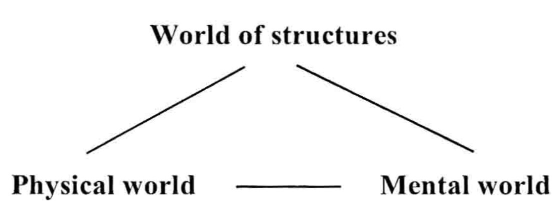

Figure 1.1. The existential triad of the world

In the physical world, there are the real tables and chairs, sun and stars, stones and flowers, space and time, molecules and atoms, electrons and photons. In the mental world, there are also real "things" and "processes". For instance, there exist happiness and pain, smell and color, love and understanding, impressions and images (of tables, chairs, stones, flowers, stars, etc.). It has been demonstrated (Burgin, 1997) that the world of structures also exists in reality. There structures as real as stars and stones, people and their cars, or as thoughts, moods and emotions.

It is necessary to understand that these three worlds are not separate realities: they interact and intersect. For instance, individual mentality is based on the brain, which is a material object. On the other hand, mentality influences physical world (cf., for example, (Herbert, 1987)).

Even closer ties exist between the structural world and material world. Actually no material thing exists without a structure. Even chaos has its chaotic structure. Structures make things what they are. For instance, it is possible to make a table from different materials: wood, plastics, copper, iron, aluminum, etc. What all these things, which are called a *table*, have in common is not their material substance; it is specific peculiarities of their structure. As physicists argue, physics studies not physical systems as they are but structures of these systems, or physical structures. In some sciences, such as chemistry, and some areas of practical activity, such as engineering, structures play the leading role. For instance, the spatial structure of atoms, chemical elements, and molecules determines many properties of these chemical systems. In engineering, structures and structural analysis form a separate discipline (cf., for example, (Martin, 1999)).

Each of these three worlds from the existential triad is hierarchically organized, comprising several levels or strata. For instance, the hierarchy of the physical world goes from subatomic particles to atoms to molecules to bodies to cells to living beings and so on. On the first level of the mental world, individual mentality is situated. The second level consists of group and social mentalities, which include collective subconscience in the sense of Jung (1969) and collective intelligence in the sense of Nguen (2008).

The structure of the world projects the corresponding structure on the phenomenon of power. As a result, we have three dimensions of power:

1. *Physical power* is a possibility to change physical reality.
2. *Mental power* is a possibility to change mental reality.
3. *Information/structural power* is a possibility to change the world of structures.

These three dimensions are explicitly represented in the structure of power in different democratic countries. For instance, in the United States, there exists the following triad of power:

Congress

President ——— **Supreme Court**
with the Departments of State

Congress of the United States as the legislative power forms the structure of society and state by changing existing laws and approves new laws. Thus, it corresponds to the information/structural power.

President of the United States with the Departments of State as the executive power controls physical functioning of the state. Thus, it corresponds to the physical power.

Supreme Court of the United States with other courts in the country, as the Supreme Court extended power, applies the law. In such a way, it forms the mental level of the US citizens and society as a whole. Thus, it corresponds to the mental power. Note that courts do not take physical actions. It is done by police and other similar organizations.

There are different relations and interactions between these dimensions of power. For instance, sociologists wrote a lot about *symbolic power* (cf., for example, (Weber, 1951; Duncan, 1969; Bourdieu, 1977; 2001)). Symbolic power is the power to make people see and believe certain visions of the world rather than others. Thus, symbolic power is a kind of mental power.

At the same time, economists and sociologists have developed the concept of structural power (cf., for example, (Kanter, 1993; Helleiner, 2005; Sarai, 2008; Holden, 2009)). Their interpretations assign two meanings to the term *structural power*. The main meaning of structural power is the ability to influence environment or context, changing their structure. The concept of structural power, as the power accrued through the definition of frameworks and rules, has been effectively used by scholars to describe the political relations of the international financial order. This concept has been particularly useful in explaining the privileged position of the United States, which has historically developed out of the complex interrelationship between the United States and financial markets, both domestic and international.

Another meaning of structural power is the power that people have due to their position in a social or institutional/organizational hierarchy. This kind of power becomes a structural determinant affecting organizational behaviours and attitudes. In this case, power is obtained from the ability to access and mobilize influence, support, opportunities, information and other resources from one's status in the organization.

Computing power is a kind of structural power because computers and their networks, such as the Internet, change structures of data and knowledge in the process of their functioning. Algorithms organize and control computations, as well as communication, and thus determine structural power of computers. As a result, power of computers and their networks is represented by and depends on power of algorithms, which is the main object of study in this book.

1.2. Axiomatic Methods in Mathematics Versus Constructive Approach

> Science may be described as the art of systematic oversimplification.
>
> *Karl Popper* (1902-1994)

Mathematical methods play more and more important role in society. Mathematics is applied to a diversity of other fields. Mathematics provides a variety of methods for description, modeling, computation, reasoning, constructing etc. This extensive variety of methods is traditionally divided into two directions: constructive and axiomatic. Beginning from Euclid's "*Elements*," which present geometry as an axiomatic discipline, axiomatic methods have demonstrated their power in mathematics. As Burton (1997) writes, generation after generation regarded the "*Elements*" as the summit and crown of logic and mathematics, and its study as the best way of developing facility of exact reasoning. Abraham Lincoln at the age of forty, while still a struggling lawyer, mastered the first six books of Euclid, solely as training for his mind. Even now, in spite of the discovery of non-Euclidean geometries and improvements of the system of Euclid, "Elements" largely remains the supreme model of a book in mathematics, demonstrating the power of the axiomatic approach.

Mathematics suggests an approach for knowledge unification, namely, it is necessary to find axioms that characterize all theories in a specific area and to develop the theory in an axiomatic context. This approach worked well in a variety of mathematical fields:

- in geometry (Euclid's axioms and later Hilbert's axioms (Hilbert, 1899));
- in algebra (actually the whole algebra exists now as an axiomatic discipline);
- in set theory, according to Fraenkel and Bar-Hillel (1958), the most important directions in the axiomatic set theory are: Zermelo-Fraenkel (**ZF**), von Neumann (**VN**), Bernays-Godel (**BG**), Quine (**NF**, **ML**) and Hao Wang (Σ) axiomatic theories;
- in topology (Hausdorff's axioms (Hausdorff, 1927)), and even in such an applied area as probability theory (Kolmogorov's axioms (Kolmogorov, 1933)).

Axiomatization has been often used in physics, biology, and some other areas, such as philosophy or technology.

The axiomatic approach acquires its name from the Greek word *axioma*, which means "that which is thought fitting; decision; self-evident principle."

In general, *axioms* are statements felt weighty enough to take them without proofs and build a theory based on these statements. However, in a mathematical context, axioms usually reflect the existing mathematical practice, while in other fields, such as physics or biology, axioms represent important properties of studied objects and their models. As a result, axioms in science and scientific theories based on axioms are tested by experiments, while mathematical axioms are only studied and used for discovering new properties of mathematical structures and relations between these structures. In doing this, the axiomatic approach has demonstrated its power in many areas of mathematics.

It is assumed that Euclid was the first to introduce and extensively develop the axiomatic approach in mathematics, writing a thirteen-volume book *The Elements*. This book organized the geometry known at Euclid's time into a systematic presentation that has been used as a model for many papers, monographs and textbooks. In the book, Euclid first defines geometrical basic terms, such as point and line, then states without proof certain axioms and postulates about them that seem to be self-evident or obvious truths, and finally derives a big number of statements (theorems) from the postulates by means of deductive logic. This axiomatic method has since been adopted not only throughout mathematics but in many other fields as well. Euclid's *Elements* set the standard of rigor for nearly two thousand years of mathematics.

However, the important philosophical predecessor of Euclid was Aristotle (384-322 B.C.E.), who discussed the first principles of any demonstrative science in the *Posterior Analytics*. According to Aristotle, a demonstrative science must start from indemonstrable principles, which were common to all sciences and were called *axioms* (cf. (Mueller, 1969)). A standard example of an axiom for Aristotle is the principle that if equals be subtracted from equals, the remainders are equal.

The axiomatic approach adopted by Euclid in his *Elements* was much later embraced and further developed by the majority of mathematicians starting from the 19th century in their quest for rigor, consistency and elegance in mathematics. Ever since Euclid the axiomatic approach is at the heart of mathematics. In addition, as the axiomatic approach admits the possibility of a mixture of deductive and empirical reasoning, it has become an ideal pedagogical tool. With the advent of computers, deductive reasoning and axiomatic exposition have been delegated to computers, which performed theorem-proving, while the axiomatic approach has come to software technology and computer science (cf., for example (Floyd, 1967; Hoare, 1969; Manna, 1974)).

The close examination of the axioms and postulates of Euclidean geometry during the 19th century resulted in the realization that the logical basis of geometry was not as firm as had previously been supposed. New axiom and postulate systems were developed by various mathematicians, notably by Hilbert (1899), in geometry. These accomplishments together with the discovery of non-Euclidean geometries and development of modern algebra brought axiomatic methods to the center of the whole mathematics.

Axiomatization means the following process. Starting with a set of unambiguous statements called axioms, whose truth is assumed, one is able to deduce all the remaining propositions of the theory from these axioms using axioms of logical inference.

In an axiomatic theory, objects of study are defined purely by used axioms. This allows one to find many interpretations and thus, application of axiomatic theories. For instance, taking an axiomatic set theory, we see that in the first approximation, it is possible to treat a multitude of systems as sets. The contemporary axiomatic approach is basically the attitude that doing mathematics, it is not necessary to know what the things researchers are working with are. Only relations between these objects, rules of operation with them and their properties are important. For instance, in an axiomatic geometry, instead of knowing what a point or a line is, it is crucial to be familiar with and be able to use the axioms that describe points and lines.

It is interesting that the axiomatic approach was also used in areas that are very far from mathematics. For instance, Spinoza (1632-1677) used this approach in philosophy, developing his ethical theories and writing his book Ethics in the axiomatic form. More recently, Kunii (2004) developed an axiomatic system for cyberworlds.

In spite of all success of the axiomatic approach in mathematics, in the final quarter of the 19th, certain mathematicians started to express disapproval of the 'idealistic', non-constructive methods used in the axiomatic mathematics, insisting on the necessity to make the whole mathematics constructive.

Constructive mathematics is distinguished from its traditional counterpart, axiomatic classical mathematics, by the strict interpretation of the expression "there exists" (called in logic the *existential quantifier*) as "we can construct" and show the way how to do this. Assertions of existence should be backed up by constructions, and the properties of mathematical objects should be decidable in finitely many steps. Thus, in order to constructively work in mathematics and related areas, such as computer science and physics, it is necessary to re-interpret not only the expression "there exists" but all the logical connectives and quantifiers as instructions on how to construct a proof of the statement involving these logical expressions and how to build the objects used in this proof. As a result, we come to necessity of algorithms as constructive descriptions of various procedures and processes. As Markov observed (1954), "the entire significance for mathematics" of efforts to define algorithm more precisely would be "in connection with the problem of a constructive foundation for mathematics".

Troelstra (1991) distinguishes eight principal trends in constructive mathematics: intuitionism, semi-intuitionism, finitism, predicativism, actualism or strict intuitionism, Markov's constructivism, recursive analysis, and Bishop's constructivism.

The first mathematician who laid the foundations of a precise, systematic approach to constructive mathematics was L.E.J. Brouwer (1881-1966). He created a philosophy and theory, known as intuitionism, according to which mathematics is a free creation of the human mind, and an object exists if and only if it can be (mentally) constructed.

With respect to mathematics, Brouwer's main ideas are

1. Mathematics is not formal, i.e., the objects of mathematics are mental constructions in the mind of the (ideal) mathematician and only such constructions are exact.
2. Mathematics is independent of experience in the outside world, as well as of language. Communication by language may serve to suggest similar thought constructions to others, but there is no guarantee that these other constructions are the same.
3. Mathematics is not based on logic. On the contrary, logic is a part of mathematics.

The term *semi-intuitionism* or *empiricism* refers to philosophical and methodological ideas of a group of French mathematicians, which included E. Borel (1871–1970), H. Lebesgue (1875–1941), R. Baire (1874-1932), and the Russian mathematician N.N. Luzin (1893-1950). Their discussions of foundational problems from mathematics were aimed at exclusion of too abstract objects, being always in direct connection with specific mathematical developments. What the semi-intuitionists have in common is the idea that, even if mathematical objects exist independently of the human mind, mathematics can only deal with such objects if mathematicians can mentally construct them.

A.A. Markov (1903-1979) formulated in 1948-49 the basic ideas of his approach, which can be named *constructive recursive mathematics*:

1. Objects of constructive mathematics are constructive, being words in various alphabets.
2. The abstraction of potential existence is admissible but the abstraction of actual infinity is not allowed.
3. A precise notion of algorithm is taken as a basis.
4. Logically compound statements have to be interpreted so as to take the meaning of component statements into account.

For this theory, Markov elaborated a special model (construction) of algorithms, which are now called normal Markov algorithms.