Cambridge studies in advanced mathematics 121

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The Okounkov–Vershik Approach, Character Formulas, and Partition Algebras

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521118170

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Ceccherini-Silberstein, Tullio.
Representation theory of the symmetric groups : the Okounkov–Vershik approach,
character formulas, and partition algebras / Tullio Ceccherini-Silberstein,
Fabio Scarabotti, Filippo Tolli.
p. cm. – (Cambridge studies in advanced mathematics ; 121)
Includes bibliographical references and index.
ISBN 978-0-521-11817-0 (hardback)
1. Symmetry groups. 2. Representations of groups. I. Scarabotti, Fabio.
II. Tolli, Filippo, 1968– III. Title. IV. Series.
QA176.C43 2010
512′.2 – dc22 2009044129

ISBN 978-0-521-11817-0 Hardback

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REPRESENTATION THEORY OF THE SYMMETRIC GROUPS

The representation theory of the symmetric groups is a classical topic that, since the pioneering work of Frobenius, Schur and Young, has grown into a huge body of theory, with many important connections to other areas of mathematics and physics.

This self-contained book provides a detailed introduction to the subject, covering classical topics such as the Littlewood–Richardson rule and the Schur–Weyl duality. Importantly, the authors also present many recent advances in the area, including M. Lassalle's character formulas, the theory of partition algebras, and an exhaustive exposition of the approach developed by A. M. Vershik and A. Okounkov.

A wealth of examples and exercises makes this an ideal textbook for graduate students. It will also serve as a useful reference for more experienced researchers across a range of areas, including algebra, computer science, statistical mechanics and theoretical physics.

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To Katiuscia, Giacomo, and Tommaso

To my parents, Cristina, and Nadiya

To my Mom, Rossella, and Stefania

Preface

Since the pioneering works of Frobenius, Schur and Young more than a hundred years ago, the representation theory of the finite symmetric group has grown into a huge body of theory, with many important and deep connections to the representation theory of other groups and algebras as well as with fruitful relations to other areas of mathematics and physics. In this monograph, we present the representation theory of the symmetric group along the new lines developed by several authors, in particular by A. M. Vershik, G. I. Olshanskii and A. Okounkov. The tools/ingredients of this new approach are either completely new, or were not fully understood in their whole importance by previous authors. Such tools/ingredients, that in our book are presented in a fully detailed and exhaustive exposition, are:

- the algebras of conjugacy-invariant functions, the algebras of bi-K-invariant functions, the Gelfand pairs and their spherical functions;
- the Gelfand–Tsetlin algebras and their corresponding bases;
- the branching diagrams, the associated posets and the content of a tableau;
- the Young–Jucys–Murphy elements and their spectral analysis;
- the characters of the symmetric group viewed as spherical functions.

The first chapter is an introduction to the representation theory of finite groups. The second chapter contains a detailed discussion of the algebras of conjugacy-invariant functions and their relations with Gelfand pairs and Gelfand–Tsetlin bases. In the third chapter, which constitutes the core of the whole book, we present an exposition of the Okounkov–Vershik approach to the representation theory of the symmetric group. We closely follow the original sources. However, we enlighten the presentation by establishing a connection between the algebras of conjugacy-invariant functions and Gelfand pairs, and by deducing the Young rule from the analysis of a suitable poset.

We also derive, in an original way, the Pieri rule. In the fourth chapter we present the theory of symmetric functions focusing on their relations with the representation theory of the symmetric group. We have added some nonstandard material, closely related to the subject. In particular, we present two proofs of the Jucys–Murphy theorem which characterizes the center of the group algebra of the symmetric group as the algebra of symmetric polynomials in the Jucys–Murphy elements. The first proof is the original one given by Murphy, while the second one, due to A. Garsia, also provides an explicit expression for the characters of $\mathfrak{S}_n$ as symmetric polynomials in the Jucys–Murphy elements. In the fifth chapter we give some recent formulas by Lassalle and Corteel–Goupil–Schaeffer. In these formulas, the characters of the symmetric group, viewed as spherical functions, are expressed as symmetric functions on the content of the tableaux, or, alternatively, as shifted symmetric functions (a concept introduced by Olshanskii and Okounkov) on the partitions. Chapter 6 is entirely dedicated to the Littlewood–Richardson rule and is based on G. D. James' approach. The combinatorial theory developed by James is extremely powerful and, besides giving a proof of the Littlewood–Richardson rule, provides explicit orthogonal decompositions of the Young modules. We show that the decompositions obtained in Chapter 3 (via the Gelfand–Tsetlin bases) are particular cases of those obtained with James' method and, following Sternberg, we interpret such decompositions in terms of Radon transforms (P. Diaconis also alluded to this idea in his book [26]). Moreover, we introduce the Specht modules and the generalized Specht modules. It is important to point out that this part is closely related to the theory developed in Chapter 3 starting from the branching rule and the elementary notions on Young modules (in fact these notions and the related results suffice). The seventh chapter is an introduction to finite dimensional algebras and their representation theory. In order to avoid technicalities and to get as fast as possible to the fundamental results, we limit ourselves to the operator $*$-algebras on a finite dimensional Hilbert space. We have included a detailed account on reciprocity laws based on recent ideas of R. Howe and their exposition in the book by Goodman–Wallach, and a related abstract construction that naturally leads to the notion of partition algebra. In Chapter 8 we present an exposition of the Schur–Weyl duality emphasizing the connections with the results from Chapters 3 and 4. We do not go deeply into the representation theory of the general linear group $GL(n, \mathbb{R})$, because it requires tools like Lie algebras, but we include an elementary account on partition algebras, mainly based on a recent expository paper of T. Halverson and A. Ram.

The style of our book is the following. We explicitly want to remain at an elementary level, without introducing the notions in their wider generality and avoiding too many technicalities. On the other hand, the book is absolutely self-contained (apart from the elementary notions of linear algebra and group theory, including group actions) and the proofs are presented in full details. Our goal is to introduce the (possibly inexperienced) reader to an active area of research, with a text that is, therefore, far from being a simple compilation of papers and other books. Indeed, in several places, our treatment is original, even for a few elementary facts. Just to draw a comparison against two other books, the theory of Okounkov and Vershik is treated in a complete way in the first chapter of Kleshchev's book, but this monograph is at an extremely more advanced level than ours. Also, the theory of symmetric functions is masterly and remarkably treated in the classical book by Macdonald; in comparison with this book, by which we were inspired at several stages, our treatment is slightly more elementary and less algebraic. However, we present many recent results not included in Macdonald's book.

We express our deep gratitude to Alexei Borodin, Adriano Garsia, Andrei Okounkov, Grigori Olshanski, and especially to Arun Ram and Anatoly Vershik, for their interest in our work, useful comments and continuous encouragement.

We also thank Roger Astley, Clare Dennison and Charlotte Broom from Cambridge University Press and Jon Billam for their constant and kindest help at all stages of the editing process.

Roma, 21 May 2009 TCS, FS and FT

Contents

1

Representation theory of finite groups

This chapter is a basic course in representation theory of finite groups. It is inspired by the books by Serre [109], Simon [111], Sternberg [115], Fulton and Harris [43] and by our recent [20]. With respect to the latter, we do not separate the elementary and the advanced topics (Chapter 3 and Chapter 9 therein). Here, the advanced topics are introduced as soon as possible.

The presentation of the character theory is based on the book by Fulton and Harris [43], while the section on induced representations is inspired by the books by Serre [109], Bump [15], Sternberg [115] and by our expository paper [18].

1.1 Basic facts

1.1.1 Representations

Let G be a finite group and V a finite dimensional vector space over the complex field $\mathbb{C}$. We denote by $GL(V)$ the group of all bijective linear maps $T : V \to V$. A (linear) *representation* of G on V is a homomorphism

$$\sigma : G \to GL(V).$$

This means that for every $g \in G$, $\sigma(g)$ is a linear bijection of V into itself and that

- $\sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2)$ for all $g_1, g_2 \in G$;
- $\sigma(1_G) = I_V$, where 1_G is the identity of G and I_V the identity map on V;
- $\sigma(g^{-1}) = \sigma(g)^{-1}$ for all $g \in G$.

To emphasize the role of V, a representation will be also denoted by the pair (σ, V) or simply by V. Note that a representation may be also seen as an action of G on V such that $\sigma(g)$ is a linear map for all $g \in G$.

A subspace $W \leq V$ is *σ-invariant* (or *G-invariant*) if $\sigma(g)w \in W$ for all $g \in G$ and $w \in W$. Clearly, setting $\rho(g) = \sigma(g)|_W$, then (ρ, W) is also a representation of G. We say that ρ is a *sub-representation* of σ.

The trivial subspaces V and $\{0\}$ are always invariant. We say that (σ, V) is *irreducible* if V has no non-trivial invariant subspaces; otherwise we say that it is *reducible*.

Suppose now that V is a *unitary* space, that is, it is endowed with a Hermitian scalar product $\langle \cdot, \cdot \rangle_V$. A representation (σ, V) is *unitary* provided that $\sigma(g)$ is a unitary operator for all $g \in G$. This means that $\langle \sigma(g)v_1, \sigma(g)v_2 \rangle_V = \langle v_1, v_2 \rangle_V$ for all $g \in G$ and $v_1, v_2 \in V$. In particular, $\sigma(g^{-1})$ equals $\sigma(g)^*$, the *adjoint* of $\sigma(g)$.

Let (σ, V) be a representation of G and let $K \leq G$ be a subgroup. The *restriction* of σ from G to K, denoted by $\mathrm{Res}^G_K \sigma$ (or $\mathrm{Res}^G_K V$) is the representation of K on V defined by $[\mathrm{Res}^G_K \sigma](k) = \sigma(k)$ for all $k \in K$.

1.1.2 Examples

Example 1.1.1 (The trivial representation) For every group G, we define the *trivial representation* as the one-dimensional representation $(\iota_G, \mathbb{C})$ defined by setting $\iota_G(g) = 1$, for all $g \in G$.

Example 1.1.2 (Permutation representation (homogeneous space)) Suppose that G acts on a finite set X; for $g \in G$ and $x \in X$ denote by gx the g-image of x. Denote by $L(X)$ the vector space of all complex-valued functions defined on X. Then we can define a representation λ of G on $L(X)$ by setting

$$[\lambda(g)f](x) = f(g^{-1}x)$$

for all $g \in G$, $f \in L(X)$ and $x \in X$. This is indeed a representation:

$$[\lambda(g_1g_2)f](x) = f(g_2^{-1}g_1^{-1}x) = [\lambda(g_2)f](g_1^{-1}x) = \{\lambda(g_1)[\lambda(g_2)f]\}(x),$$

that is, $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$ (and clearly $\lambda(1_G) = I_{L(X)}$). λ is called the *permutation representation* of G on $L(X)$.

If we introduce a scalar product $\langle \cdot, \cdot \rangle_{L(X)}$ on $L(X)$ by setting

$$\langle f_1, f_2 \rangle_{L(X)} = \sum_{x \in X} f_1(x)\overline{f_2(x)}$$

for all $f_1, f_2 \in L(X)$, then λ is unitary.