# calculus: a short course

richmond

# CALCULUS: A SHORT COURSE

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*For Mary*

# PREFACE

In revising my earlier book, *Introductory Calculus,* I have attempted to make the subject applicable to problems in the social and behavioral sciences as well as in biology, business, and economics. The fact that recent years have seen more students in these disciplines taking an introductory calculus course has guided my efforts in preparing this edition.

This text is intended as the basis for a semester course in calculus. It presupposes no knowledge of analytic geometry or trigonometry. It may therefore be taught during the freshman year, in conjunction with a semester course in finite mathematics or in statistics, to students with a background of three years of secondary mathematics or, after a semester of algebra, to students with less background.

The content of elementary calculus has been carefully rethought with this purpose in mind. The first chapter deals with functions (mostly rational) and their graphs, as well as the idea of continuity. Every effort has been made to introduce the limit concept in a clear, intuitive way. And in the present edition, the proof of the Limit Theorem has been recast and the treatment of sequences which become infinite has been improved. Many minor changes have been made. Sufficient material is included so that all theorems may be proved rigorously. On the other hand, this development may be broken off at various points, depending upon the maturity of the student.

Chapter 2 introduces the analytic geometry of a straight line and treats the tangent to a curve as the best linear approximation to it in the neighborhood of the point of contact. The treatment of the derivative as the slope of the linear approximation makes an abbreviation of the proofs of the differentiation theorems possible and gives them a certain concreteness. This is notably the case with the Chain Rule. By using closed intervals and assuming Weierstrass' Theorem (existence of maxima and minima on a closed interval), it is possible to give a simple and complete treatment of maxima and minima and to study the shape of a graph near a given point. A natural treatment of the Mean Value Theorem is given along the same lines. The chapter closes with applications and rate problems, and a section on applications to business and economics.

Chapter 3 is devoted to the idea of the area under a curve, which is characterized by three requirements. Assuming that the area function exists, it is shown that it is uniquely defined by these properties and that its derivative is the ordinate. After the determination of areas bounded by curves, the chapter concludes with a definition of area in terms of summation and an existence proof for the area function.

Chapter 4 is a study of the exponential functions defined by $y = 2^x$ and $y = e^x$. All essential properties are derived, and it is shown how values of the exponential may be calculated by using facts about the area below its graph. The proof that $y = 2^x$ is concave upward has been simplified. The natural logarithm is introduced as the inverse of the exponential. Applications to radioactive decay and free fall with air resistance have been retained from the previous edition; and new material has been added on retarded growth, population studies, and learning theory.

Chapter 5 contains a development of analytic trigonometry by way of complex numbers. Let $x$ be the arc length measured along the unit circle from its point of intersection with the real axis. Then $\cos x$ and $\sin x$ are defined as the real and imaginary parts of the corresponding complex number. The addition and subtraction formulas follow immediately, as do the differentiation formulas for the sine and cosine. Graphs of $\sin x$ and $\cos x$ are easily constructed. A method is developed for computing values of these functions for arbitrary $x$, by iteration, and this process is proved to converge. The other trigonometric functions are briefly treated, and $\pi$ is calculated from the integral

$$\int_0^1 \frac{dx}{1 + x^2} .$$

Two new sections have been added on the solution of $D_t^2x + k^2x = 0$ and its applications, especially to the equilibrium of species.

An effort has been made to develop each chapter around a central idea and to emphasize the nature of mathematical thinking. It has been the author's conviction that mathematics texts suffer from a lack of plot. It is hoped that the present text will not be subject to this criticism and that its use will give some feeling for mathematical proofs. The number of exercises has been substantially increased in this edition. Many improvements in detail reflect the experience of my colleagues and helpful comments made by those who have used the book elsewhere.

This book is intended for a semester course. For a longer course, the author's *Calculus with Analytic Geometry* may be used. It contains the five chapters of this book and the following additional chapters: "Antidifferentiation and Integration," "Definite Integrals. Applications," "Linear Differential Equations," "Vectors," "The Inverse Square Law," "Vectors in Space. Partial Derivatives," "Multiple Integration. Volumes," "Approximation of Functions. Series."

*Stanford, Calif.*
*June 1969*

D. E. R.

# INTRODUCTION

Each of the following chapters is concerned with a single central idea. In each case, this idea is at first rather vague and inexact. It must be sharpened to a precise *definition.* Once this is done, it is possible to build on the definition a body of theory which may be summarized in a number of proved statements or *theorems.* These theorems will be found to answer almost automatically many of the questions which come to mind in connection with the central idea from which we started. Hence we are rewarded for doing some fundamental thinking by the discovery of procedures which may be used almost without thinking.

It is hoped that this method of organization will bring out the nature of mathematical thought so that the student will acquire a feeling for the way in which mathematics develops. It is an all too prevalent opinion that mathematics consists of a set of rules or routines to be carried out mechanically. There is of course some truth in this opinion, since, as we have said, the end result of a theoretical development is often the establishment of procedures which *are* almost automatic. This has its good and bad sides. It is good because mathematics has applications in all aspects of the natural sciences and technology and more recently in the social sciences. In the everyday practice of these subjects, it is important that after a certain amount of training, people may learn to employ certain mathematical techniques in a semiautomatic way, releasing thought (which is difficult) for the occasions where it is necessary. The unfortunate side of this situation is that it tends to convey the impression that the subject is a rather dull one in which there is no creative element. This is the opposite of the case. In fact, there is no subject which, correctly understood, makes greater demands upon the imagination. To extract from an intuitive situation the conceptional essence and pin it down so that it forms the basis of a fruitful structure requires the highest order of creative intelligence. The beautiful way in which ideas connect with each other makes many parts of mathematics into artistically satisfying pieces of logical architecture. The great mathematician works in a spirit akin to that of a musical composer. It is hoped that the student will get something of this feeling from the study of these chapters.

It is not necessary for him to be a genius to do so. It *is* necessary that he study the subject with emphasis upon the organization of ideas, rather than upon getting quick answers to numerical problems, that he attempt to understand the way in which the argument builds up. To do so, requires frequent re-reading and reflection. It is not enough to study one section at a time and work out a few problems.

Even from the most practical point of view, it is important to study mathematics in this way. All techniques have their limitations and one who does not know what lies behind them will use them blindly and therefore unintelligently. Moreover it is unlikely indeed that in a changing world, any set of routine operations will prove to be adequate to all of the situations which will arise in practice. If one cannot make some modifications to suit the circumstances, he will be handicapped indeed. The student is therefore urged to bring his own creative talents to bear on the subject and to acquire a feeling of freedom about it. By doing so, he will obtain pleasure from the study of mathematics and fit himself to contribute badly needed understanding to the solution of the problems of our society.

It is the contention of most informed thinkers that mathematics has only begun to make its proper contribution to modern life, and that totally new fields of application are to be anticipated. This we believe to be true. However, the speed with which these applications will be made will certainly depend upon the soundness of mathematical education. Those in a position to make significant new uses of mathematics will miss their opportunities unless they have a feeling for *mathematical thinking* in addition to a knowledge of certain routines.

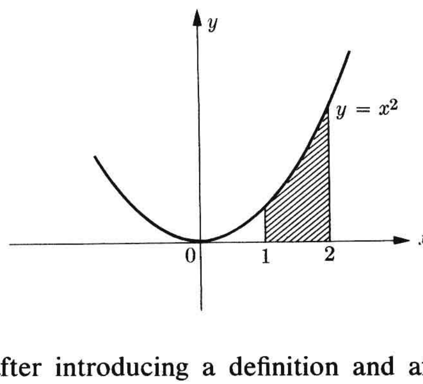
One further general remark is in order. The student will soon discover that there is a remarkable parallelism between statements which may be made about geometrical figures and diagrams on the one hand and equations and formulas on the other hand—in a word, between geometry and algebra. More accurately, in the calculus we deal with a kind of extension of algebra called *analysis,* in which extra symbols are introduced, such as $\rightarrow$, the symbol for *approaches.* The point is that our proofs will look like algebra and, like algebra, the variables $x$, $y$, $z$ will indicate places where *numbers* may be fitted in. It may well be asked why we use this symbolism in proofs, when some of the statements proved correspond to facts which are obvious geometrically, that is, on the diagram.

There are several reasons for this. One is that analytic proofs, as they are called, are found to give greater security and certainty. It is easy to overlook certain possibilities on the picture. For example, not until late in the 19th century was it realized, by the German mathematician Weierstrass, that functions could exist whose graphs were continuous (without breaks) but so wiggly that no piece, however short, could be approximated by a straight line. These curves have no direction at any point. Their discovery was made analytically and was a great surprise geometrically.

Actually, mathematics owes much of its fruitfulness to the interplay between geometrical intuition and analytic proof. Our intuition suggests what is likely to be true. In the attempt to prove this analytically we may well discover as we manipulate our symbols that to justify a step, we need to assume something which we had previously overlooked. On the other hand, we may notice that the "algebra" seems to work all right without some of the restrictions which we had in mind. We then look back to the figure and ask what meaning it has to remove these restrictions.

There is a second and important reason for preferring analytic proofs besides their greater security. In working with symbols, we tend to free ourselves from the pictures which we had in our minds. It then happens very often indeed that the symbolic structure or theoretical scheme has applications to a great many other things than we originally intended. Mathematics is abstract in the sense that a body of theory has many different possible interpretations in terms of intuitive content. The same body may be dressed up in many different suits of clothes.

This point will become clearer as we proceed. By way of illustration, in Chapter 3 we start with the intuitive idea of area under a portion of a graph.




In due course, after introducing a definition and after translating our geometrical ideas into analytic terms (referring to the *equation* of the graph and so on), we develop a surprisingly simple method of finding such areas as that in the figure. Then we notice that the procedure has many interpretations other than area, such as the distance covered by a body moving with the velocity $v = t^2$ from the time when $t = 1$ sec to that when $t = 2$ sec, or the work done in moving a body a certain distance against a certain force, and so on.

Mathematics is concerned with the logical consequences of different sets of assumptions. The consequences are true whenever the assumptions are true. It is fortunate that assumptions of the same type may be made in a wide variety of circumstances, so that a theory built with one interpretation in mind may be transferred bodily to another field of application. This is one of the reasons why mathematics has a virtually unlimited scope. The discovery of the applicability of a worked-out logical structure to a new field may produce revolutionary advances. The development of such structures is therefore not only exciting to the mathematician, but of great promise to the understanding of the world in which we live.

# CONTENTS

CHAPTER 1

# FUNCTIONS AND GRAPHS

## 1–1 FUNCTIONS AND GRAPHS

The basic idea of this chapter is that of a function. Imagine a machine which is so constructed that for each number fed into it (punched on a tape) a single number comes out. The relation of the output number to the input number is determined by the construction of the machine. For example, the machine might be a number *squarer.* Then if we put in the number 2, the number 4 will come out. If we put in 3, 9 will come out. Associated with each input number is a corresponding output number called its square.

Schematically, we may write

$$\text{Output} = (\text{Input})^2. \tag{1}$$

It is customary, however, to use the *variables* $x$ and $y$ and write

$$y = x^2. \tag{2}$$

There is nothing mysterious about these letters. They represent places into which numbers can be put. If we fill these places at random, (2) will not hold. Thus if we substitute $x = 1$ and $y = 2$, (2) is not satisfied, since 2 is not the square of 1. If $x = 4$ and $y = 16$, (2) *is* satisfied. Thus Eq. (2) defines a *pairing* of numbers ($x$) and their squares ($x^2$). When (2) holds we say that $y$ is the square of $x$. The equation $y = x^2$ is said to define or determine a certain *function* of $x$.

This is an example of an important idea, the concept of a function, for which we give a formal definition.

**Definition.** A *function* ($f$) is a correspondence between two sets of numbers, called the *domain* of the function and the *range* of the function, such that with every number $x$ of the domain there is associated exactly one number $y$ of the range. This $y$ is called the *value* of the function for the given $x$.

The function is said to be *defined* for all the numbers which are in its domain and for no others.

In calculus the association is usually specified by an equation which gives the $y$ that corresponds to a given $x$ as a formula involving $x$. This

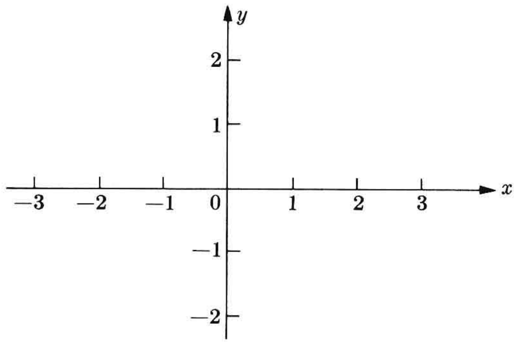



**Figure 1–1**

was the case in our example, where $y = x^2$ gives the $y$ to be associated with a given $x$. Unless the contrary is stated, it is understood that the domain consists of all numbers $x$ for which the formula gives a value $y$. In the case of $y = x^2$, the domain consists of all *real* numbers, a term which will soon be explained. The range consists of 0 and the positive real numbers.

To represent the correspondence of $x$ with $y$ geometrically, we introduce a pair of axes. By providing each axis with a scale, values of $x$ are associated with points on a horizontal line called the $x$-axis and values of $y$ with points on a vertical line called the $y$-axis (Fig. 1–1). Thus on the $x$-axis we mark a point with the label 0 (zero) and a point to the right of it with the label 1. Using the distance from 0 to 1 as a unit of measure, we locate points labeled successively 2, 3, 4, and so on. By applying this unit of measure to the left of 0, we obtain the points labeled $-1, -2, -3, \ldots$. The points so located are called *integral* points, since the numbers used to name them are the *integers*, positive $1, 2, 3, 4, \ldots$, negative $-1, -2, -3, \ldots$, and zero 0.

Every point on the $x$-axis may be represented by a decimal, ending or unending. Thus 1.32 represents a point between 1 and 2 found by dividing this interval into 100 equal subintervals and counting off 32 of them. The point $\frac{1}{3}$ of the way from 0 to 1 is represented by the unending decimal $0.333\ldots$. The aggregate of all possible decimals, ending or unending, is called the set of *real numbers*. Further examples of real numbers are 0.25, $-1.111\ldots$, $3.14159\ldots (= \pi)$, and $2.0\ (= 2)$. Every point on the $x$-axis corresponds to some real number and, conversely, every real number represents a point on the $x$-axis. A fuller discussion of real numbers and scales may be found in the Appendix.

We provide the $y$-axis with a scale in a similar manner, making the 0 of the $y$-axis agree with that of the $x$-axis. This common point, marked $O$, or 0, is called the *origin*. We also agree to take the point 1 on the $y$-axis *above* the origin so that the positive direction on the $y$-axis is the upward

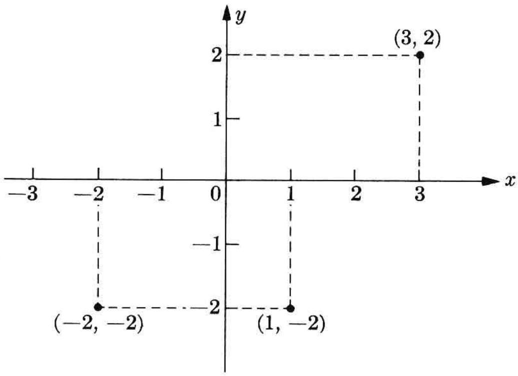



**Figure 1–2**

one, the negative direction, the downward one. It is not necessary that the unit [01] on the $y$-axis have the same length as that on the $x$-axis, but we shall assume for the present that it has.

Once the axes have been drawn and marked off, we are in a position to represent the correspondence of a given $y$ with a given $x$. We draw a vertical line (a line parallel to the $y$-axis) through the point on the $x$-axis which corresponds to the given $x$, and a horizontal line (parallel to the $x$-axis) through the point on the $y$-axis representing the corresponding $y$. The point $P$ of intersection of these two lines shall represent the correspondence of this $x$ and this $y$. $x$ and $y$ are called the *coordinates* of $P$; $x$ is the *abscissa* and $y$ the *ordinate.* The notation $(a, b)$ will be used to denote the point whose abscissa is $a$ and whose ordinate is $b$. The points (3, 2), $(-2, -2)$, and $(1, -2)$ are shown in Fig. 1–2.

After these preliminaries, we can proceed to the *graph* of $y = x^2$, which consists of all points whose coordinates satisfy (2). We prepare a short table of values of $x$ and $y$ $(= x^2)$:

| $x$ | $y\ (= x^2)$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| −1 | 1 |
| −2 | 4 |
| −3 | 9 |

and locate the corresponding points, $O, A, B, C, D, E, F$ (Fig. 1–3).

We now sketch a curve passing through the plotted points, as indicated in the figure. The graph then gives a geometrical picture of the relation

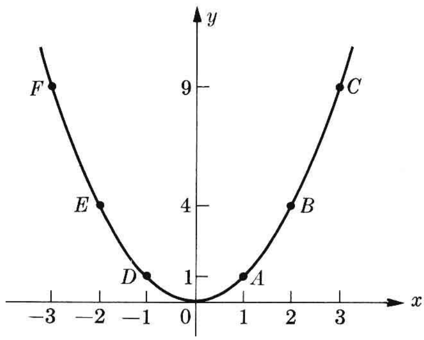



**Figure 1–3**

$y = x^2$, in the sense that all pairs of numbers which satisfy this equation are the coordinates of points on the curve, and all points on the curve have coordinates $x$ and $y$ for which $y = x^2$. The connection between algebra and the geometry of the plane brought about by the use of axes and scales is a very fruitful one. We may visualize a function in terms of its graph and study a graph by means of the relation between the $x$- and $y$-coordinates of its points. This is the general program of this chapter.

The following observations concerning this program can be made at the present stage:

1. On a graph of a function of $x$, there can be only one $y$ for a given $x$. This means that the graph of a function cannot have one point directly above or below another. (There may very well be several $x$'s for a single $y$, as in Fig. 1–3.)

2. The domain of definition specified by $y = x^2$ consists of all real numbers. The example $y = \sqrt{x}$ shows that it may be necessary to exclude some real numbers from the domain of a function, since $\sqrt{x}$ has no real value if $x$ is negative. (If $y = \sqrt{x}$, $x = y^2 \geq 0$.) It is important, then, to specify the domain of definition of a given function.

3. In sketching the curve in our example, we assumed that nothing very exciting could happen between the plotted points so that, for example, the curve would have no breaks in it nor would it develop a high peak between two points. This assumption requires investigation. Some functions do have breaks in their graphs. Others do not. How can we determine what will happen? We can always plot more points, but we cannot plot all possible points. A new principle is needed. For the moment, however, we must use our intuition to sketch the graphs of a few functions and thus gain experience.

**EXERCISES**

Let $y$ be given in terms of $x$ by the following expressions. Plot the corresponding graphs.

1. $y = x^3$
2. $y = x - 3$
3. $y = \dfrac{3x + 1}{2}$
4. $y = \dfrac{x^2}{2}$
5. $y = x^2 - 2x$
6. $y = x^4$
7. $y = x^3 - 6x$
8. $y = x^3 - 6x^2 + 9x$
9. $y = 3$
10. $y = 2x^2$
11. $y = (x - 1)^2$
12. $y = -2x + 3$
13. $y = \dfrac{1}{x - 1}$
14. $y = \sqrt{x}$
15. $y = \sqrt[3]{x}$
16. $y = \dfrac{1}{x^2}$
17. $y = \dfrac{1}{x}$
18. $y = \sqrt{x - 2}$

## 1–2 GRAPHS

The examples so far considered have been rather simple. Except for the last few exercises, we have been concerned with so-called polynomials, for which each $y$-value is built up from the corresponding $x$-value by taking powers of $x$ (for example $x^2$, $x^3$, $x^4$), multiplying by constants, and adding or subtracting the resulting terms. Complications occur when we extend our operations to include division and root extraction.

**Example 1.** Consider the function defined by the equation

$$y = \frac{1}{x - 2}.$$

The formula $1/(x - 2)$ defines no value of $y$ when $x = 2$, since $1/0$ is a meaningless symbol. (Division by zero is impossible.) All other $x$-values do lead to $y$-values. The domain of this function therefore consists of the set of all values of $x$ different from 2. We denote this domain by the symbol $x \neq 2$.

The graph is shown in Fig. 1–4. We note that for values of $x$ slightly greater than 2, $y$ is large and positive, while for $x$-values slightly less than 2, $y$ is negative and numerically large. To study the situation near $x = 2$, it is convenient to consider the values of $y$ for the succession of $x$-values given by $x = 3, 2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots$ and by $x = 1, 1\frac{1}{2}, 1\frac{2}{3}, 1\frac{3}{4}, \ldots$. In general, we take $x = x_n = 2 + (1/n)$ and $x_n = 2 - (1/n)$, $n = 1, 2, 3, \ldots$. Thus $y = 1000$ when $x = 2\frac{1}{1000}$, $y = 1{,}000{,}000$ when $x = 2\frac{1}{1{,}000{,}000}$, and so on. We can find a point on the graph as high as we please by choosing an $x$-value near enough to 2 and to its right. Similarly, we can find points on the graph below any specified level by choosing $x$ close enough to 2 on the left. We describe this situation by saying that $y$ becomes *positively infinite* as $x$ approaches 2 from the right and $y$ becomes *negatively infinite* as $x$ approaches