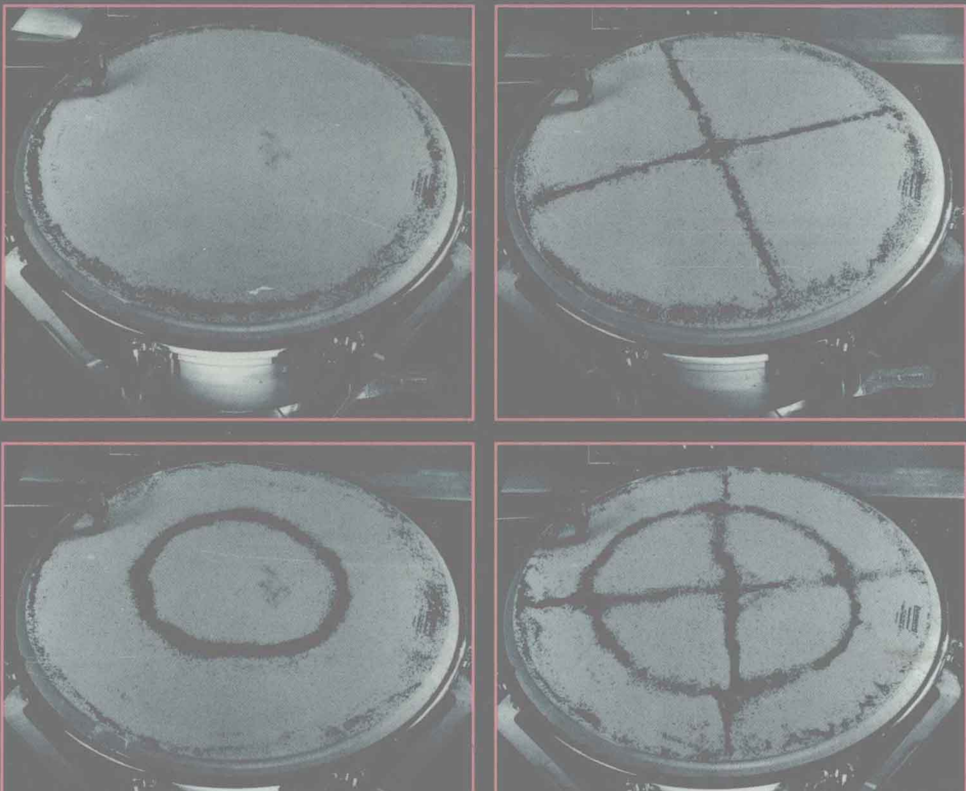
Boundary Value Problems



David L. Powers

Third Edition

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PREFACE

This text is designed for a one-semester or two-quarter course in partial differential equations given to third- and fourth-year students of engineering and science. It may also be used as the basis for an introductory course for graduate students. Mathematical prerequisites have been kept to a minimum—calculus and differential equations. No vector calculus or linear algebra (other than 2 × 2 determinants) is necessary. The reader is assumed to have enough background in physics to follow the derivations of the heat and wave equations.

The principal objective of the book is solving boundary value problems involving partial differential equations. Separation of variables receives the greatest attention because it is widely used in applications and because it provides a uniform method for solving important cases of the heat, wave, and potential equations. One technique is not enough, of course. D'Alembert's solution of the wave equation is developed in parallel with the series solution, and the distributed-source solution is constructed for the heat equation. In addition, there are chapters on Laplace transform techniques and on numerical methods.

The secondary objective is to tie together the mathematics developed and the student's physical intuition. This is accomplished by deriving the mathematical model in a number of cases, by using physical reasoning in the mathematical development from time to time, by interpreting mathematical results in physical terms, and by studying the heat, wave, and potential equations separately.

In the service of both objectives, there are many fully worked examples and over 750 exercises, including miscellaneous exercises at the end of each chapter. The level of difficulty ranges from drill and verification of details to development of new material. Answers to odd-numbered exercises are in the back of the book.

There are many ways of choosing and arranging topics from the book so as to provide an interesting and meaningful course. The following sections form the core, requiring at least 14 hours of lecture: Chapter 1, Sections 1–3; Chapter 2, Sections 1–5; Chapter 3, Sections 1–3; Chapter 4, Sections 1, 2, and 4. These cover the basics of Fourier series and the solutions of heat, wave, and potential equations in finite regions. My choice for the next most important block of material is the Fourier integral and the solution of problems on unbounded regions: Chapter 1, Section 9; Chapter 2, Sections 10 and 11; Chapter 3, Section 6; Chapter 4, Section 3. These require at least six more lectures.

The taste of the instructor and the needs of the audience will govern the choice of further material. A rather theoretical flavor results from including: Chapter 1, Sections 4–7 on Fourier series; Chapter 2, Sections 7–9 on Sturm-Liouville problems, and the sequel, Chapter 3, Section 4; and the more difficult parts of Chapter 5, Sections 5–9 on Bessel functions and Legendre polynomials. On the other hand, inclusion of Chapter 7, Numerical Methods, gives a very applied flavor, especially if students write programs and run them on a computer.

Chapter 0 reviews solution techniques and theory of ordinary differential equations and boundary value problems. Equilibrium forms of the heat and wave equations are derived also. This material belongs in an elementary differential equations course and is strictly optional. However, many students have either forgotten it or never seen it.

In this Third Edition, many sections have been rewritten. Section 5 of Chapter 0 on Green's functions and Section 7 of Chapter 1, a Fourier series convergence proof, are both new. Chapters 0 and 7 were extensively reorganized. Three short BASIC program segments are included for Fourier coefficients, Gauss-Jordan elimination, and Gauss-Seidel iteration. Finally, some 200 new exercises have been added, and various useful mathematical formulas are collected in a new appendix.

Now it is my pleasure to thank publicly some friends and colleagues. Victor Lovass-Nagy and Abdul J. Jerri have suggested numerous improvements and additions over the entire life of the book. Advice, encouragement, suggestions and corrections have come from many colleagues and students including Heino Ainso, Charles Cullen, James Foster, Charles Haines, M. M. Ibrahim, Charles Marshall, Gustave Rabson, Hayley Shen, Harvey Segur and Kin Wah Tse. The photographs on the cover were kindly supplied by Professor T. D. Rossing.

Finally, I wish to acknowledge the assistance of reviewers Ilya Bakelman of Texas A & M University, William Royalty of the University of Idaho, Al Shenk of the University of California at San Diego, Michael W. Smiley of Iowa State University, and Monty J. Strauss of Texas Tech University.

David L. Powers

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CHAPTER 0

ORDINARY DIFFERENTIAL EQUATIONS

1. HOMOGENEOUS LINEAR EQUATIONS

The subject of most of this book is partial differential equations: their physical meaning, problems in which they appear, and their solutions. Our principal solution technique will involve separating a partial differential equation into ordinary differential equations. Therefore, we begin by reviewing some facts about ordinary differential equations and their solutions.

We are interested mainly in linear differential equations of first and second orders, as shown in Eqs. (1) and (2):

$$\frac{du}{dt} = k(t)u + f(t), \tag{1}$$

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = f(t). \tag{2}$$

In either equation, if $f(t)$ is 0, the equation is homogeneous. (Another test: if the constant function $u(t) \equiv 0$ is a solution, the equation is homogeneous.) In the rest of this section, we review homogeneous linear equations.

A. FIRST-ORDER EQUATIONS

The most general first-order homogeneous equation has the form

$$\frac{du}{dt} = k(t)u. \tag{3}$$

This equation can be solved by isolating u on one side and then integrating:

$$\frac{1}{u}\frac{du}{dt} = k(t)$$

$$\ln|u| = \int k(t)dt + C$$

$$u(t) = \pm e^{C}e^{\int k(t)dt} = ce^{\int k(t)dt} \tag{4}$$

It is easy to check directly that the last expression is a solution of the differential equation for any value of c. That is, c is an arbitrary constant and can be used to satisfy an initial condition if one has been specified.

For example, let us solve the **homogeneous differential equation**

$$\frac{du}{dt} = -tu.$$

The procedure outlined above gives the general solution

$$u(t) = ce^{-t^2/2}$$

for any c. If an initial condition such as $u(0) = 5$ is specified, then c must be chosen to satisfy it ($c = 5$).

The most common case of this differential equation has $k(t) = k$ constant. The differential equation and its **general solution** are

$$\frac{du}{dt} = ku, \; u(t) = ce^{kt}. \tag{5}$$

If k is negative, then $u(t)$ approaches 0 as t increases. If k is positive, then $u(t)$ increases rapidly in magnitude with t. This kind of exponential growth often signals disaster in physical situations, since it cannot be sustained indefinitely.

B. SECOND-ORDER EQUATIONS

It is not possible to give a solution method for the general second-order linear homogeneous equation,

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = 0. \tag{6}$$

Nevertheless, we can solve some important cases that we detail below. The most important point in the general theory is the following.

Principle of Superposition. If $u_1(t)$ and $u_2(t)$ are solutions of the same linear homogeneous equation (6), then so is any linear combination of them:

$$u(t) = c_1u_1(t) + c_2u_2(t).$$

This theorem, which is very easy to prove, merits the name of *principle* because it applies, with only superficial changes, to many other kinds of linear, homogeneous equations. Later, we will be using the same principle on partial differential equations.

To be able to satisfy an unrestricted initial condition, we need two linearly independent solutions of a second-order equation. Two solutions are *linearly independent* if the only linear combination of them (with constant coefficients) that is identically 0 is the combination with 0 for its coefficients. There is an alternative test. Two solutions of the same linear homogeneous equation (6) are independent if and only if their *Wronskian*

$$W(u_1, u_2) = \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix} \tag{7}$$

is nonzero.

1. Constant coefficients. The most important type of second-order linear differential equation that can be solved in closed form is the one with constant coefficients,

$$\frac{d^2u}{dt^2} + k\frac{du}{dt} + pu = 0 \quad (k, p \text{ are constants}). \tag{8}$$

There is always at least one solution of the form $u(t) = e^{mt}$ for an appropriate constant m. To find m, substitute the proposed solution into the differential equation, obtaining

$$m^2e^{mt} + kme^{mt} + pe^{mt} = 0, \quad \text{or}$$
$$m^2 + km + p = 0 \tag{9}$$

(since e^{mt} is never 0). This is called the *characteristic polynomial* of the differential equation (8). There are three cases for the roots of the characteristic equation (9), which determine the nature of the general solution of Eq. (8). These are summarized in Table 0.1.

TABLE 0–1

SOLUTIONS OF $\dfrac{d^2u}{dt^2} + k\dfrac{du}{dt} + pu = 0$

Roots of Characteristic Polynomial	General Solution of Differential Equation
Real, distinct: $m_1 \neq m_2$	$u(t) = c_1e^{m_1t} + c_2e^{m_2t}$
Real, double: $m_1 = m_2$	$u(t) = c_1e^{m_1t} + c_2te^{m_1t}$
Conjugate complex: $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$	$u(t) = c_1e^{\alpha t}\cos \beta t + c_2e^{\alpha t}\sin \beta t$

This method of assuming an exponential form for the solution works for linear homogeneous equations of any order with constant coefficients. In all cases, a pair of complex conjugate roots $m = \alpha \pm i\beta$ lead to a pair of complex solutions

$$e^{\alpha t}e^{i\beta t},\ e^{\alpha t}e^{-i\beta t} \tag{10}$$

which can be traded for the pair of real solutions

$$e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t. \tag{11}$$

We include two important examples. First, consider the differential equation

$$\frac{d^2u}{dt^2} + \lambda^2 u = 0 \tag{12}$$

where λ is constant. The characteristic polynomial of this equation is $m^2 + \lambda^2 = 0$, with roots $m = \pm i\lambda$. The third case applies; the general solution is

$$u(t) = c_1 \cos\lambda t + c_2 \sin\lambda t. \tag{13}$$

Second, consider the similar differential equation

$$\frac{d^2u}{dt^2} - \lambda^2 u = 0. \tag{14}$$

The characteristic polynomial now is $m^2 - \lambda^2 = 0$, with roots $m = \pm\lambda$. If $\lambda > 0$, the first case applies, and the general solution is

$$u(t) = c_1 e^{\lambda t} + c_2 e^{-\lambda t}. \tag{15}$$

It is sometimes helpful to write the solution in another form. The hyperbolic sine and cosine are defined by

$$\sinh A = \frac{1}{2}(e^A - e^{-A}),\ \cosh A = \frac{1}{2}(e^A + e^{-A}). \tag{16}$$

Thus, $\sinh\lambda t$ and $\cosh\lambda t$ are linear combinations of $e^{\lambda t}$ and $e^{-\lambda t}$. By the principle of superposition, they too are solutions of Eq. (14). The Wronskian test shows them to be independent. Therefore, we may equally well write

$$u(t) = c_1' \cosh\lambda t + c_2' \sinh\lambda t$$

as the general solution of Eq. (14), where c_1' and c_2' are arbitrary constants.

2. Cauchy-Euler Equation. One of the few equations with variable coefficients that can be solved in complete generality is the Cauchy-Euler equation:

$$t^2\frac{d^2u}{dt^2} + kt\frac{du}{dt} + pu = 0. \tag{17}$$

The distinguishing feature of this equation is that the coefficient of the nth derivative is the nth power of t, multiplied by a constant. The style of solution for this equation is quite similar to the preceding: assume that a solution has the form $u(t) = t^m$, then find m. Substituting u in this form into Eq. (17) leads to

$$t^2m(m - 1)t^{m-2} + kt\,mt^{m-1} + pt^m = 0, \text{ or}$$
$$m(m - 1) + km + p = 0 \quad (k,p \text{ are constants}). \tag{18}$$

This is the characteristic polynomial for Eq. (17), and the nature of its roots determines the solution as summarized in Table 0.2.

One important example of the Cauchy-Euler equation is

$$t^2 \frac{d^2u}{dt^2} + t\frac{du}{dt} - \lambda^2 u = 0 \tag{19}$$

where $\lambda > 0$. The characteristic polynomial is $m(m - 1) + m - \lambda^2 = m^2 - \lambda^2$. The roots are $m = \pm\lambda$, so the first case of Table 0.2 applies, and

$$u(t) = c_1 t^\lambda + c_2 t^{-\lambda} \tag{20}$$

is the general solution of Eq. (19).

For the general linear equation

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = 0,$$

any point where $k(t)$ or $p(t)$ fails to be continuous is a *singular point* of the differential equation. At such a point, solutions may break down in various ways. However, if t_0 is a singular point where both of the functions

$$(t - t_0)k(t) \text{ and } (t - t_0)^2p(t) \tag{21}$$

have Taylor series expansions, then t_0 is called a *regular singular point*. The Cauchy-Euler equation is an example of an important differential equation hav-

TABLE 0–2

SOLUTIONS OF $t^2\frac{d^2u}{dt^2} + kt\frac{du}{dt} + pu = 0$

Roots of Characteristic Polynomial	General Solution of Differential Equation
Real distinct roots: $m_1 \neq m_2$	$u(t) = c_1t^{m_1} + c_2t^{m_2}$
Real double root: $m_1 = m_2$	$u(t) = c_1t^{m_1} + c_2(\ln t)t^{m_1}$
Conjugate complex roots: $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$	$u(t) = c_1t^\alpha\cos(\beta \ln t) + c_2t^\alpha\sin(\beta \ln t)$

ing a regular singular point (at $t_0 = 0$). The behavior of its solutions near that point provides a model for more general equations.

3. Other Equations. Other second-order equations may be solved by power series, by change of variable to a kind already solved, or by sheer luck. For example, the equation

$$t^4 \frac{d^2u}{dt^2} + \lambda^2 u = 0, \tag{22}$$

which occurs in the theory of beams, can be solved by the change of variables

$$t = \frac{1}{z}, \quad u(t) = \frac{1}{z} v(z).$$

In terms of the new variables, the differential equation (22) becomes

$$\frac{d^2v}{dz^2} + \lambda^2 v = 0.$$

This equation is easily solved, and the solution of the original is then found by reversing the change of variables:

$$u(t) = t\,(c_1 \cos(\lambda/t) + c_2 \sin(\lambda/t)). \tag{23}$$

C. SECOND INDEPENDENT SOLUTION

Although it is not generally possible to solve a second-order linear homogeneous equation with variable coefficients, we can always find a second independent solution if one nontrivial solution is known.

Suppose $u_1(t)$ is a solution of the general equation

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = 0. \tag{24}$$

Assume that $u_2(t) = v(t)u_1(t)$ is a solution. We wish to find $v(t)$ so that u_2 is indeed a solution. However, $v(t)$ must not be constant, since that would not supply an independent solution. A straightforward substitution of $u_2 = vu_1$ into the differential equation leads to

$$v''u_1 + 2v'u_1' + vu_1'' + k(t)\,(v'u_1 + vu_1') + p(t)vu_1 = 0.$$

Now collect terms in the derivatives of v. The equation above becomes

$$u_1v'' + (2u_1' + k(t)u_1)v' + (u_1'' + k(t)u_1' + p(t)u_1)v = 0.$$

However, u_1 is a solution of Eq. (24), so the coefficient of v is 0. This leaves

$$u_1 v'' + (2u_1' + k(t)u_1)v' = 0, \tag{25}$$

which is a first-order linear equation for v'. Thus, a nonconstant v can be found, at least in terms of some integrals.

For example, consider the equation

$$(1 - t^2)u'' - 2tu' + 2u = 0,$$

which has $u_1(t) = t$ as a solution. By assuming that $u_2 = v \cdot t$ and substituting, we get

$$(1 - t^2)(v''t + 2v') - 2t(v't + v) + 2vt = 0.$$

After collecting terms, we have

$$(1 - t^2)tv'' + (2 - 4t^2)v' = 0.$$

From here, it is fairly easy to find

$$\frac{v''}{v'} = \frac{4t^2 - 2}{t(1 - t^2)} = \frac{-2}{t} + \frac{1}{1 - t} - \frac{1}{1 + t}$$

(using partial fractions), then

$$\ln v' = -2 \ln t - \ln(1 - t) - \ln(1 + t).$$

Finally, each side is exponentiated to get

$$v' = \frac{1}{t^2(1 - t^2)} = \frac{1}{t^2} + \frac{1}{1 - t^2}$$

$$v = -\frac{1}{t} + \frac{1}{2} \ln \left| \frac{1 + t}{1 - t} \right|.$$

This is a nonconstant v, so it provides a second independent solution:

$$u_2(t) = vt = -1 + \frac{1}{2} t \ln \left| \frac{1 + t}{1 - t} \right|.$$

Summary Some important equations and their solutions follow.

1. $\dfrac{du}{dt} = ku$ (k is constant)

$u(t) = ce^{kt}$

2. $\dfrac{d^2u}{dt^2} + \lambda^2 u = 0$

$u(t) = a \cos \lambda t + b \sin \lambda t$