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Konrad Schmüdgen

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To Katja and Alexander

Les théories ont leurs commencements: des allusions vagues, des essais inachevés, des problèmes particuliers; et même lorsque ces commencements importent peu dans l'état actuel de la Science, on aurait tort de les passer sous silence.

F. Riesz,
Les systèmes d'équations
linéaires a une infinité
d'inconnues,
Paris, 1913, p. 1.

Scientific subjects do not progress necessarily on the lines of direct usefulness. Very many applications of the theories of pure mathematics have come many years, sometimes centuries, after the actual discoveries themselves. The weapons were at hand, but the men were not able to use them.

A. R. Forsyth,
Perry's Teaching of Mathematics,
London, 1902, p. 35.

Preface

$*$-algebras of unbounded operators in Hilbert space, or more generally algebraic systems of unbounded operators, occur in a natural way in unitary representation theory of Lie groups and in the Wightman formulation of quantum field theory. In representation theory they appear as the images of the associated representations of the Lie algebras or of the enveloping algebras on the Garding domain and in quantum field theory they occur as the vector space of field operators or the $*$-algebra generated by them. Some of the basic tools for the general theory were first introduced and used in these fields. For instance, the notion of the weak (bounded) commutant which plays a fundamental role in the general theory had already appeared in quantum field theory early in the sixties. Nevertheless, a systematic study of unbounded operator algebras began only at the beginning of the seventies. It was initiated by (in alphabetic order) BORCHERS, LASSNER, POWERS, UHLMANN and VASILIEV. From the very beginning, and still today, representation theory of Lie groups and Lie algebras and quantum field theory have been primary sources of motivation and also of examples. However, the general theory of unbounded operator algebras has also had points of contact with several other disciplines. In particular, the theory of locally convex spaces, the theory of von Neumann algebras, distribution theory, single operator theory, the moment problem and its non-commutative generalizations and noncommutative probability theory, all have interacted with our subject.

This book is an attempt to provide a treatmant of $*$-algebras of unbounded operators in Hilbert space (the so-called O*-algebras) and of (unbounded) $*$-representations of general $*$-algebras. Roughly speaking, an *O*-algebra* is a $*$-algebra $\mathcal{A}$ of linear operators defined on a common dense linear subspace $\mathcal{D}$ of a Hilbert space and leaving $\mathcal{D}$ invariant. The multiplication in $\mathcal{A}$ is the composition of operators, which makes sense because of the invariance of the domain $\mathcal{D}$, and the involution $a \to a^+$ in $\mathcal{A}$ is defined by letting a^+ be the restriction to $\mathcal{D}$ of the usual Hilbert space adjoint a^*. We always assume that an O*-algebra on $\mathcal{D}$ contains the identity map of $\mathcal{D}$. A *$*$-representation* of a general $*$-algebra with unit is a $*$-homomorphism of the $*$-algebra onto some O*-algebra. Moreover, we also consider some more general families of closable linear operators (O-families, O-vector spaces, O-algebras, O*-families and O*-vector spaces) which are always defined on a common dense domain $\mathcal{D}$.

Our objective is threefold. First, the book gives a thorough treatment of certain of the basic concepts involved in the theory of O*-algebras and $*$-representations. These mainly concern notions like the graph topology, closed and self-adjoint $*$-representations, closed and self-adjoint O*-algebras, weak and strong (bounded) commutants, strongly positive and completely strongly positive $*$-representations, to name the most important, which have proved to be useful and fundamental in the theory. We also develop concepts like directed O-families, commutatively dominated O*-algebras, weak and strong unbounded commutants, form commutants, induced extensions and strongly n-positive $*$-representations with the anticipation that these will be useful in future research. Secondly, we aim to prove some of the more involved results of the existing theory. As a sample, results in Sections 2.4, 4.3, 5.3, 5.4, 6.2, 7.3, 9.2, 9.4, 10.2, 10.4, 10.5, 11.2, 12.3 and 12.4 could be mentioned in this respect. Thirdly, the book presents many examples and counter-examples that help to delimit the general theory. These sometimes require more involved constructions and arguments than many of the positive results in the theory. For instance, we construct a self-adjoint $*$-representation of the polynomial algebra in two variables, the bounded commutant of which is a given properly infinite von Neumann algebra in separable Hilbert space.

The scope of this book is, of course, dictated by the stage of the existing theory. Thus, for instance, the topological theory of O*-algebras occupies a relatively large space in this monograph, simply because it is much more developed than other parts of the theory. The choice of the material contained in this book also depends on the author's personal view of the existing theory and on his particular research interests. Some topics such as GB*-algebras, Hilbert algebras, tensor algebras and applications in physics are not included. Often the original proofs of the results have been improved, errors have been corrected or the result has been generalized. Frequently the terminology and the notation have been changed, we hope for the better. Also several new concepts are introduced.

Apart from the preliminary chapter, the book consists of two parts which are independent to a large extent (see also the introduction to Part II). In Part I O*-algebras and topologies on the domains and the algebras are studied, while Part II is concerned with $*$-representations of general $*$-algebras. Those topics in the theory of $*$-representations that primarily involve the study of topologies or the structure of O*-algebras are treated in part I. Such topics are the continuity of $*$-representations, the realization of the generalized Calkin algebra and the abstract characterization of the $*$-algebras $\mathcal{L}^+(\mathcal{D}_i : i \in I)$. Chapter 10 gives a rather thorough treatment of integrable representations of Lie algebras resp. enveloping algebras. This chapter stands almost entirely by itself; it requires only a few general definitions and facts from earlier sections.

Almost no bibliographical comments are given in the body of the text; they are gathered in a section entitled "Notes" at the end of each chapter. There, the sources of the main results, basic concepts and some examples are cited (of course, as far as the author is aware), but no attempt has been made to be encyclopaedic. Some of these sections contain a list of references dealing with problems similar to those in the text.

The first two digits in the number of a theorem, proposition, lemma, definition or example refer to the section and the third digit to the position of the item. Remarks and formulas are numbered and quoted consecutively within the sections. When a reference to a formula in another section is made, the number of the section is added; for instance, 3.2/(1) means formula (1) in Section 3.2. The end of a proof is marked by □ and of an example by ○. The reader should also note that we often fix assumptions or notations at the beginning of a chapter, section or subsection which keep in force throughout the whole chapter, section or subsection. Further, the proofs of facts stated in the examples are frequently merely sketched and sometimes they are omitted altogether.

I am grateful to Dr. Jürgen Friedrich and Dr. Klaus-Detlef Kürsten for their critical reading of large parts of the manuscript and for many valuable suggestions. I am also very indebted to Professor Paul S. Muhly for his help in writing this book. Last but not least, I wish to thank R. Helle, Dr. R. Höppner and G. Reiher of the Akademie-Verlag for their patience and help in preparing this book.

Leipzig, Fall 1987 K. Schmüdgen

Contents

1. Preliminaries

In this chapter we summarize some basic definitions, notation and results that will be required in this monograph. Some, but not all, of them are standard or well known. General terminology which is used essentially in one chapter, section or subsection will be introduced therein.

First we collect some general notation. Throughout, $\mathbb{C}$ denotes the complex numbers, $\mathbb{T}$ the complex numbers of modulus one, $\mathbb{R}$ the real numbers, $\mathbb{Z}$ the integers, $\mathbb{N}$ the positive integers and $\mathbb{N}_0$ the non-negative integers. For $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ and $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$, t^n is the usual multi-index notation, i.e., $t^n := t_1^{n_1} \ldots t_d^{n_d}$, where $t_k^0 := 1$ for $k = 1, \ldots, d$. The abbreviations l.h. and c.l.h. mean the linear hull and the closed linear hull, respectively. Sequences and nets are written as $(x_n : n \in \mathbb{N})$ resp. $(x_i : i \in I)$ or simply as (x_n) resp. (x_i). In general, sets are denoted by braces such as $\{x_n : n \in \mathbb{N}\}$. For an open or closed subset M of $\mathbb{R}^d$, $L^p(M)$ is the L^p-space with respect to the Lebesgue measure on M. If M is a C^∞-manifold (with or without boundary) and $n \in \mathbb{N} \cup \{\infty\}$, then $C^n(M)$ is the set of all complex functions of class C^n on M. We denote by $C_0^\infty(M)$ the set of all functions in $C^\infty(M)$ whose support is a compact subset of M. The continuous complex functions on a topological space M are denoted by $C(M)$. For a and b in $\mathbb{R}$, we shall write $C^\infty[a, b]$ for $C^\infty([a, b])$, $C[a, b]$ for $C([a, b])$, $C_0^\infty(a, b)$ for $C_0^\infty\big((a, b)\big)$ and $L^p(a, b)$ for $L^p\big((a, b)\big)$. As usual, δ_{nm} is the Kronecker symbol. The closed unit ball of a normed space E is denoted by $\mathcal{U}_E$.

1.1. Locally Convex Spaces

As general references for the theory of locally convex spaces we shall use the textbooks Schäfer [1], Köthe [1], [2] and Jarchow [1].

All considered vector spaces are either over the real field $\mathbb{R}$ or over the complex field $\mathbb{C}$. When we speak about a vector space or a locally convex space without specifying the field, we always mean spaces over $\mathbb{C}$. Let U and M be subsets of a vector space E over $\mathbb{K}$. Then U *absorbs* M if there is an $\alpha > 0$ such that $M \subseteq \lambda U$ for all $\lambda \in \mathbb{K}$, $|\lambda| \geqq \alpha$, and U is *absorbing* if it absorbs every singleton $\{\varphi\}$, $\varphi \in E$. The absolutely convex hull of U is denoted by aco U.

If τ is a topology on a set E, then we write $E[\tau]$ for the corresponding topological space. The induced topology on a subset F of E is denoted by $\tau \restriction F$ or simply again by τ if no confusion can arise. If τ_1 and τ_2 are topologies on E, then $\tau_1 \subseteq \tau_2$ means that τ_1 is coarser (weaker) than τ_2.

A *locally convex space* is a (not necessarily Hausdorff) topological vector space over $\mathbb{K} = \mathbb{R}$ or over $\mathbb{K} = \mathbb{C}$ which has a 0-neighbourhood base $\boldsymbol{U}$ satisfying the following conditions:

(i) For $U_1, U_2 \in \boldsymbol{U}$, there is a $U \in \boldsymbol{U}$ such that $U \subseteqq U_1 \cap U_2$.

(ii) If $U \in \boldsymbol{U}$, then $\lambda U \in \boldsymbol{U}$ for all $\lambda \in \mathbb{K}$, $\lambda \neq 0$.

(iii) Each $U \in \boldsymbol{U}$ is absolutely convex and absorbing.

If $\boldsymbol{U}$ is a non-empty family of subsets of a real or complex vector space which fulfills (i), (ii) and (iii), then there is a unique topology τ on E such that $E[\tau]$ is a locally convex space and $\boldsymbol{U}$ is a 0-neighbourhood base for τ. By a *locally convex topology* on a vector space E we mean a topology τ on E for which $E[\tau]$ is a locally convex space. Let Γ be a non-empty family of seminorms on a vector space E. The collection $\boldsymbol{U}$ of all sets

$$\{\varphi \in E : p_n(\varphi) \leqq \varepsilon \text{ for } n = 1, \ldots, k\}, \text{ where } p_1, \ldots, p_k \in \Gamma,\ k \in \mathbb{N} \text{ and } \varepsilon > 0,$$

satisfies (i)—(iii); so $\boldsymbol{U}$ is a 0-neighbourhood base for a unique locally convex topology τ on E. We then say that τ is *generated* (or *defined* or *determined*) by Γ. The family Γ is *directed* if, given $p_1, p_2 \in \Gamma$, there is a $p \in \Gamma$ such that $p_1 \leqq p$ and $p_2 \leqq p$.

In what follows we suppose that E is a locally convex Hausdorff space.

Let E' denote the dual of E. The *weak topology* $\sigma \equiv \sigma(E, E')$ is the locally convex topology on E defined by the seminorms $\varphi \to |\varphi'(\varphi)|$, $\varphi' \in E'$. A sequence $(\varphi_n : n \in \mathbb{N})$ in E *converges weakly* to $\varphi \in E$ if it converges in the locally convex space $E[\sigma]$ to φ, i.e. if $\lim_n \varphi'(\varphi_n) = \varphi'(\varphi)$ for all $\varphi' \in E'$. The *weak*-topology* $\sigma' \equiv \sigma(E', E)$ on E' is generated by the seminorms $\varphi' \to |\varphi'(\varphi)|$, $\varphi \in E$. The *strong topology* on E' is denoted by β; it is generated by the family of seminorms

$$r_M(\varphi') := \sup_{\varphi \in M} |\varphi'(\varphi)|, \quad \varphi' \in E',$$

where M ranges over the bounded subsets of E. The vector space E becomes a linear subspace of $(E'[\beta])'$ by identifying $\varphi \in E$ with the linear functional $\varphi' \to \varphi'(\varphi)$ on E'. E is *semireflexive* if $E = (E'[\beta])'$ under this identification, and E is *reflexive* if E is semireflexive and if the topology of E coincides with the strong topology on $(E'[\beta])'$.

A *Frechet space* is a complete metrizable locally convex space. The locally convex space E is said to be a *quasi-Frechet space* (or briefly, a *QF-space*) if for every bounded set M in E there is a subspace G of E which is a Frechet space in the induced topology of E and which contains M. It is obvious that each Frechet space is a QF-space.

The space E is *barrelled* if every barrel in E (i.e., every closed absolutely convex absorbing subset of E) is a 0-neighbourhood in E. E is a *semi-Montel space* if each bounded subset of E is relatively compact. A *Montel space* is a barrelled semi-Montel space. The space E is *bornological* if every absolutely convex set in E that absorbs each bounded set in E is a 0-neighbourhood of E. The *bornological topology associated with the topology of E* is the coarsest bornological topology on E which is finer than the topology of E.

A *fundamental system* of bounded sets in E is a family S of bounded sets such that every bounded subset of E is contained in some set of S. The space E is a *DF-space* if it admits a countable fundamental system of bounded sets and if it has the following property: If the intersection of a sequence of closed absolutely convex 0-neighbourhoods in E absorbs all bounded sets, then it is itself a 0-neighbourhood in E.

A *precompact set* in E is a set which is relatively compact in the completion of E. (Often these sets are called *totally bounded*.)

Lemma 1.1.1. *Let Γ be a directed family of seminorms which generates the topology of E, and let M be a subset of E. Suppose that, given $p \in \Gamma$ and $\varepsilon > 0$, there exists a bounded set $M_{p,\varepsilon}$ contained in a finite dimensional subspace of E such that for each $\varphi \in M$ there is a $\psi \in M_{p,\varepsilon}$ satisfying $p(\varphi - \psi) \leqq \varepsilon$. Then M is a precompact set in E.*

Proof. Without loss of generality we can assume that E is already complete and $M_{p,\varepsilon}$ is closed. We have to show that the closure $\overline{M}$ of M is compact. For let W be an ultrafilter on M. Fix $p \in \Gamma$ and $\varepsilon > 0$. Set $V := \{\varphi \in E : p(\varphi) \leqq \varepsilon\}$. The set $M_{p,\varepsilon}$ is compact, so there exists a finite set N in E such that $M_{p,\varepsilon} \subseteqq N + V$. By assumption, $M \subseteqq M_{p,\varepsilon} + V$. The set $M_{p,\varepsilon} + V$ is closed in E (because $M_{p,\varepsilon}$ is compact). Hence $\overline{M} \subseteqq M_{p,\varepsilon} + V \subseteqq (N + V) + V = N + 2V = \bigcup_{\psi \in N} (\psi + 2V)$. Because W is an ultrafilter, this implies that $(\psi + 2V) \in W$ for some $\psi \in N$. Since $(\psi + 2V) - (\psi + 2V) = 4V$ and the sets $4V$ form a 0-neighbourhood base on E, this shows that W is a Cauchy filter on E. Hence W is convergent and M is compact. □

The locally convex space E admits the *approximation property* if the identity map of E can be approximated, uniformly on every precompact subset of E, by continuous linear mappings of finite rank.

Suppose that the topology of E is generated by a directed family, say Γ, of norms on E. Then E is called a *Schwartz space* if for every $p \in \Gamma$ there is a $q \in \Gamma$ such that the set $\{\varphi \in E : q(\varphi) \leqq 1\}$ is precompact in the normed linear space (E, p).

Let E and F be locally convex Hausdorff spaces. We define the two main topologies on the algebraic tensor product $E \otimes F$ of E and F. For seminorms p and q on E and F, respectively, let $p \otimes_\pi q$ denote the seminorm on $E \otimes F$ which is defined by

$$p \otimes_\pi q(z) = \inf \left\{ \sum_{n=1}^{k} p(\varphi_n)\, q(\psi_n) \right\}, \quad z \in E \otimes F,$$

where the infimum is taken over all representations $z = \sum_{n=1}^{k} \varphi_n \otimes \psi_n$ in $E \otimes F$. Suppose Γ_E and Γ_F are directed families of seminorms which generate the topologies of E and F, respectively. The *projective tensor topology* on $E \otimes F$ is defined by the family of seminorms $\{p \otimes_\pi q : p \in \Gamma_E \text{ and } q \in \Gamma_F\}$. Equipped with it, the space $E \otimes F$ is called the *projective tensor product* and denoted by $E \otimes_\pi F$. Let $E \mathbin{\widehat{\otimes}_\pi} F$ be the completion of $E \otimes_\pi F$. We denote by $\mathfrak{E}(E)$ and $\mathfrak{E}(F)$ the equicontinuous subsets of E' and F', respectively. For $M \in \mathfrak{E}(E)$ and $N \in \mathfrak{E}(F)$, let

$$\varepsilon_{M,N}(z) := \sup_{\varphi' \in M} \sup_{\psi' \in N} \left| \sum_{n=1}^{k} \varphi'(\varphi_n)\, \psi'(\psi_n) \right|, \quad z = \sum_{n=1}^{k} \varphi_n \otimes \psi_n \in E.$$

The *injective tensor topology* is generated by the family of seminorms $\{\varepsilon_{M,N} : M \in \mathfrak{E}(E)$ and $N \in \mathfrak{E}(F)\}$. The injective tensor product $E \otimes_\varepsilon F$ is the vector space $E \otimes F$ endowed with this topology. The completion of $E \otimes_\varepsilon F$ is denoted by $E \mathbin{\widehat{\otimes}_\varepsilon} F$.

The following result is occasionally called the *Mittag-Leffler theorem*.

Lemma 1.1.2. *Let $(E_n : n \in \mathbb{N}_0)$ be a sequence of Banach spaces. Suppose that for each $n \in \mathbb{N}_0$, E_{n+1} is a dense linear subspace of E_n and the embedding map of E_{n+1} into E_n is continuous. Then $E_\infty := \bigcap_{n \in \mathbb{N}_0} E_n$ is dense in each space E_k, $k \in \mathbb{N}_0$.*

Proof. There is no loss of generality to assume that $k = 0$. Suppose $\varphi \in E_0$ and $\varepsilon > 0$. Let $\|\cdot\|_n$, $n \in \mathbb{N}_0$, denote the norm of E_n. Since the embedding of E_{n+1} into E_n is con tinuous, there exists a constant $\alpha_n > 0$ such that $\|\cdot\|_n \leqq \alpha_n \|\cdot\|_{n+1}$ on E_{n+1} for $n \in \mathbb{N}_0$. Upon replacing $\|\cdot\|_n$ by $\alpha_1\alpha_2 \dots \alpha_{n-1} \|\cdot\|_n$ for $n \in \mathbb{N}$, we can assume without loss of generality that $\|\cdot\|_n \leqq \|\cdot\|_{n+1}$ for $n \in \mathbb{N}_0$. Set $\varphi_0 := \varphi$. Since E_{n+1} is dense in E_n, we can construct inductively a sequence $(\varphi_n : n \in \mathbb{N}_0)$ of elements $\varphi_n \in E_n$ such that $\|\varphi_{n+1} - \varphi_n\|_{n+1} \leqq \varepsilon 2^{-n-1}$ for $n \in \mathbb{N}_0$. Then we have

$$\|\varphi_{m+n+r} - \varphi_{m+n}\|_m \leqq \sum_{l=1}^{r} \|\varphi_{m+n+l} - \varphi_{m+n+l-1}\|_m \leqq$$
$$\sum_{l=1}^{r} \|\varphi_{m+n+l} - \varphi_{m+n+l-1}\|_{m+n+l} \leqq \sum_{l=1}^{r} \varepsilon 2^{-m-n-l} < \varepsilon 2^{-n} \tag{1}$$

for $m, n \in \mathbb{N}_0$ and $r \in \mathbb{N}$. From this we conclude that the sequence $(\varphi_{m+n} : n \in \mathbb{N}_0)$ is a Cauchy sequence in the Banach space E_m, $m \in \mathbb{N}_0$. Let ψ denote the limit of the sequence $(\varphi_{0+n} : n \in \mathbb{N}_0)$ in E_0. Then, of course, ψ is also the limit of $(\varphi_{m+n} : n \in \mathbb{N}_0)$ in E_m for all $m \in \mathbb{N}_0$. Hence $\psi \in E_\infty$. Setting $m = n = 0$ and letting $r \to +\infty$ in (1), we obtain $\|\psi - \varphi\|_0 \leqq \varepsilon$ which shows that E_∞ is dense in E_0. □

1.2. Spaces of Linear Mappings and Spaces of Sesquilinear Forms

First let E and F be vector spaces. We denote by E^- the complex conjugate vector space of E. That is, E^- is equal to E as a set, the addition in E^- is the same as in E, but the multiplication by scalars is replaced in E^- by the mapping $(\lambda, \varphi) \to \bar{\lambda}\varphi$, $\lambda \in \mathbb{C}$ and $\varphi \in E$. Let $L(E, F)$ be the vector space of all linear mappings of E into F, and let $B(E, F)$ denote the vector space of all sesquilinear forms on $E \times F$. We set $L(E) := L(E, E)$ and $B(E) := B(E, E)$. A *sesquilinear form* on $E \times F$ is a mapping of $E \times F$ into $\mathbb{C}$ which is linear in the first and conjugate linear in the second variable. For $\mathfrak{c} \in B(E, F)$, define $\mathfrak{c}^+(\psi, \varphi) := \overline{\mathfrak{c}(\varphi, \psi)}$, $\varphi \in E$ and $\psi \in F$; then $\mathfrak{c}^+ \in B(F, E)$. If $\mathfrak{c} \in B(E, E)$ and $\varphi, \psi \in E$, then we have the so-called *polarization identity*

$$4\mathfrak{c}(\varphi, \psi) = \mathfrak{c}(\varphi + \psi, \varphi + \psi) - \mathfrak{c}(\varphi - \psi, \varphi - \psi) + \mathrm{i}\mathfrak{c}(\varphi + \mathrm{i}\psi, \varphi + \mathrm{i}\psi)$$
$$- \mathrm{i}\mathfrak{c}(\varphi - \mathrm{i}\psi, \varphi - \mathrm{i}\psi). \tag{1}$$

It is proved by computing the right-hand side of (1).

From now on we assume in this section that E and F are locally convex spaces. Since the vector spaces E and E^- have the same convex sets, they have the same locally convex topologies. We also denote by E^- the vector space E^- equipped with the topology of E. We shall write E^+ for the conjugate vector space $(E')^-$ of the dual E' of E. Let $\mathfrak{L}(E, F)$ denote the vector space of continuous linear mappings of E into F. Set $\mathfrak{L}(E) := \mathfrak{L}(E, E)$. A sesquilinear form $\mathfrak{c}$ on $E \times F$ is said to be *separately continuous* if $\overline{\mathfrak{c}(\varphi, \cdot)} \in F'$ for each $\varphi \in E$ and $\mathfrak{c}(\cdot, \psi) \in E'$ for each $\psi \in F$; $\mathfrak{c}$ is called *continuous* if it is a continuous mapping of $E \times F$ into $\mathbb{C}$, when $E \times F$ carries the product topology. We denote the vector spaces of all separately continuous sesquilinear forms and of all continuous sesquilinear forms by $\mathfrak{B}(E, F)$ and $\mathscr{B}(E, F)$, respectively. From the theory of locally convex spaces (see SCHÄFER [1], III, 5.1) it is known that $\mathfrak{B}(E, F) = \mathscr{B}(E, F)$ if E and F are Frechet spaces or if E and F are barrelled (DF)-spaces.