The Mechanics and Thermodynamics of Continua

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THE MECHANICS AND THERMODYNAMICS OF CONTINUA

The Mechanics and Thermodynamics of Continua presents a unified treatment of continuum mechanics and thermodynamics that emphasizes the universal status of the basic balances and the entropy imbalance. These laws are viewed as fundamental building blocks on which to frame theories of material behavior. As a valuable reference source, this book presents a detailed and complete treatment of continuum mechanics and thermodynamics for graduates and advanced undergraduates in engineering, physics, and mathematics. The chapters on plasticity discuss the standard isotropic theories and crystal plasticity and gradient plasticity.

Morton E. Gurtin is the Alumni Professor Emeritus of Mathematics at Carnegie Mellon University. His research concerns nonlinear continuum mechanics and thermodynamics, with recent emphasis on applications to problems in materials science. Among his many awards are the 2004 Timoshenko Medal of the American Society of Mechanical Engineers (ASME) "in recognition of distinguished contributions to the field of applied mechanics"; the Agostinelli Prize (an annual prize in pure and applied mathematics and mathematical physics); Accademia Nazionale dei Lincei, Italy; Dottore Honoris Causa, Civil Engineering, University of Rome; Distinguished Graduate School Alumnus Award, Brown University; and the Richard Moore Education Award, Carnegie Mellon University. In addition to his numerous archival research publications, Professor Gurtin is the author of *Configurational Forces as Basic Concepts in Continuum Physics*, *An Introduction to Continuum Mechanics*, *Thermomechanics of Evolving Phase Boundaries in the Plane*, *Topics in Finite Elasticity*, *The Linear Theory of Elasticity*, *Handbuch der Physik*, Volume VIa/2, and *Wave Propagation in Dissipative Materials* (with B. D. Coleman, I. Herrera, and C. Truesdell).

Eliot Fried is a Professor of Mechanical Engineering at McGill University, where he holds the Tier I Canada Research Chair in Interfacial and Defect Mechanics. His research focuses on the mechanics and thermodynamics of novel materials, including liquid crystals, surfactant solutions, hydrogels, granular materials, biovesicles, and nanocrystalline alloys. He is the recipient of an NSF Mathematical Sciences Postdoctoral Fellowship, a Japan Society for the Promotion of Science Postdoctoral Research Fellowship, and an NSF Research Initiation Award. Prior to joining McGill, he held tenured faculty positions in the Department of Theoretical and Applied Mechanics at the University of Illinois at Urbana-Champaign and in the Department of Mechanical, Aerospace and Structural Engineering at Washington University in St. Louis. At Illinois, he was a Fellow of the Center of Advanced Study and was awarded a Critical Research Initiative Grant. His current research is funded by the National Science Foundation, the Department of Energy, the National Institute of Health, the Natural Sciences and Engineering Research Council of Canada, the Canada Research Chairs Program, and the Canada Foundation for Innovation.

Lallit Anand is the Rohsenow Professor of Mechanical Engineering at MIT. He has had more than twenty-five years of experience conducting research and teaching at MIT, as well as substantial research experience in industry. In 1975 he began his career as a Research Scientist in the Mechanical Sciences Division of the Fundamental Research Laboratory of U.S. Steel Corporation, and he joined the MIT faculty in 1982. His research concerns continuum mechanics of solids, with focus on inelastic deformation and failure of engineering materials. In 1992, Anand was awarded the Eric Reissner Medal for "outstanding contributions to the field of mechanics of materials" from the International Society for Computational Engineering Science. In 2007 he received the Khan International Medal for "outstanding lifelong contributions to the field of plasticity" from the *International Journal of Plasticity*. He is also a Fellow of the ASME.

Preface

The Central Thrust of This Book

A large class of theories in continuum physics takes as its starting point the balance laws for mass, for linear and angular momenta, and for energy, together with an entropy imbalance that represents the second law of thermodynamics. Unfortunately, most engineering curricula teach the momentum balance laws for an array of materials, often without informing students that these laws are actually independent of those materials. Further, while courses do discuss balance of energy, they often fail to mention the second law of thermodynamics, even though its place as a basic law for continua was carefully set forth by Truesdell and Toupin[1] almost half a century ago.

This book presents a unified treatment of continuum mechanics and thermodynamics that emphasizes the universal status of the basic balances and the entropy imbalance. These laws and an hypothesis – the principle of frame-indifference, which asserts that physical theories be independent of the observer (i.e., frame of reference) – are viewed as fundamental building blocks upon which to frame theories of material behavior.

The basic laws and the frame-indifference hypothesis – being independent of material – are common to all bodies that we discuss. On the other hand, particular materials are defined by additional equations in the form of constitutive relations (such as Fourier's law) and constraints (such as incompressibility). Trivially, such constitutive assumptions reflect the fact that two bodies, one made of steel and the other of wood, generally behave differently when subject to prescribed forces – even though the two bodies obey the same basic laws.

Our general discussion of constitutive equations is based on:

(i) the principle of frame-indifference;
(ii) the use of thermodynamics to restrict constitutive equations via a paradigm generally referred to as the *Coleman–Noll procedure*.

[1] TRUESDELL & TOUPIN (1960, p. 644). In the 1960s and early 1970s this form of the second law, generally referred to as the Clausius–Duhem inequality (cf. footnote 152), was considered to be controversial because – as the argument went – the notions of entropy and temperature make no sense outside of equilibrium, an argument that stands in stark contrast to the fact that temperatures are routinely measured at shock waves. The religious nature of this argument together with the observation that most conventional theories are consistent with this form of the second law gradually led to its general acceptance – and its overall power in describing new and more general theories gave additional credence to its place as a basic law of continuum physics.

Because frame-indifference and the Coleman–Noll procedure represent powerful tools for developing physically reasonable constitutive equations, we begin our discussion by developing such equations for:

(I) the conduction of heat in a rigid medium, as this represents an excellent vehicle for demonstrating the power of the Coleman–Noll procedure;
(II) the mechanical theories of both compressible and incompressible, linearly viscous fluids, where frame-indifference applied within a very general constitutive framework demonstrates the veracity of conventional constitutive relations for fluids.

Based on frame-indifference and using the Coleman–Noll procedure, we discuss the following topics: elastic solids under isothermal and nonisothermal conditions; coupled elastic deformation and species transport, where the species in question may be ionic, atomic, molecular, or chemical; both isotropic and crystalline plastic solids; and viscoplastic solids. In our treatment of these subjects, we consider general large-deformation theories as well as corresponding small-deformation theories.

Our discussion of rate-independent and rate-dependent plasticity is *not* traditional. Unlike – but compatible with – conventional treatments, we consider flow rules that give the deviatoric stress as a function of the plastic strain-rate (and an internal variable that represents hardening).[2] We also provide a parallel description of the conventional theory based on the *principle of virtual power*. We do this because: (i) it allows us to account separately for the stretching of the microscopic structure and the flow of dislocations through that structure as described, respectively, by the elastic and plastic strain-rates; (ii) it allows for a precise discussion of material stability; and (iii) it provides a basic structure within which one can formulate more general theories. In this last regard, conventional plasticity cannot characterize recent experimental results exhibiting size effects. To model size-dependent phenomena requires a theory of plasticity with one or more material length-scales. A number of recent theories – referred to as gradient theories – accomplish this by allowing for constitutive dependencies on gradients of plastic strain and/or its rate. Such dependencies generally lead to nonlocal flow rules in the form of partial differential equations with concomitant boundary conditions. For that reason, we find it most useful to develop gradient theories via the principle of virtual power, a paradigm that automatically delivers the partial-differential equations and boundary conditions from natural assumptions regarding the expenditure of power.

Requirements of space and pedagogy led us to omit several important topics such as liquid crystals, non-Newtonian fluids, configurational forces, relativistic continuum mechanics, computational mechanics, classical viscoelasticity, and couple-stress theory.

For Whom Is This Book Meant?

Our goal is a book suitable for engineers, physicists, and mathematicians. Moreover, with the intention of providing a valuable reference source, we have tried to present a fairly detailed and complete treatment of continuum mechanics and thermodynamics. Such an ambitious scope requires a willingness to bore some when discussing issues not familiar to others. We have used parts of this book with good

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[2] We do this for consistency with the remainder of the book, which is based on the requirement that "the stress in a body is determined by the history of the motion of that body"; cf. TRUESDELL & NOLL (1965, p. 56). When discussing crystalline bodies, the flow rules express the resolved shear on the individual slip systems in terms of corresponding slip rates.

success in teaching graduates and advanced undergraduates in engineering, physics, and mathematics.

Direct Notation

For the most part, we use direct – as opposed to component (i.e., index) – notation. While some engineers and physicists might find this difficult, at least at first, we believe that the gain in clarity and insight more than compensates for the initial effort required. For those not familiar with direct notation, we have included helpful sections on vector and tensor algebra and analysis, and we present the most important results in both direct and component form.

Rigor

We present careful proofs of the basic theorems of the subject. However, when the proofs are complicated or lengthy they generally appear in petite at the end of the section in question. We also do not normally state smoothness hypotheses. Indeed, standard differentiability assumptions sufficient to make an argument rigorous are generally obvious to mathematicians and of little interest to engineers and physicists.

Attributions and Historical Issues

Our emphasis is on basic concepts and central results, not on the history of our subject. For correct references before 1965, we refer the reader to the great encyclopedic handbook articles of TRUESDELL & TOUPIN (1960) and TRUESDELL & NOLL (1965). These articles do not discuss plasticity; for the early history of that subject we refer the reader to the books of HILL (1950) and MALVERN (1969). For more recent work, we attempted to cite the contributions most central to our presentation, and we apologize in advance if we have not done so faultlessly.

Our Debt

We owe much to the chief cultivators of continuum mechanics and thermodynamics whose great work during the years 1947–1965 led to a rennaisance of the field. Their names, listed chronologically with respect to their earliest published contributions, are Ronald Rivlin, Clifford Truesdell, Jerald Ericksen, Richard Toupin, Walter Noll, and Bernard Coleman. With the exception of plasticity theory, much of this book stems from the work of these scholars – work central to the development of a unified treatment of continuum mechanics and thermodynamics based on (a) a precise statement of the balance laws for mass, linear and angular momentum, and energy, together with an entropy imbalance (the Clausius–Duhem inequality) that represents the second law of thermodynamics; (b) the unambiguous distinction between these basis laws and the notion of constitutive assumptions; and (c) a clear and compelling statement of material frame-indifference.

We are grateful to Paolo Podio-Guidugli, Guy Genin, and Giuseppe Tomassetti for their many valuable comments concerning the section on plasticity; to B. Daya Reddy for his help in developing material on variational inequalities for plasticity; and to Ian Murdoch for extensive discussions that expanded our understanding of the frame-indifference principle. Others who have contributed to this work are Paolo Cermelli, Xuemei Chen, Shaun Sellers, and Oleg Shklyaev.

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