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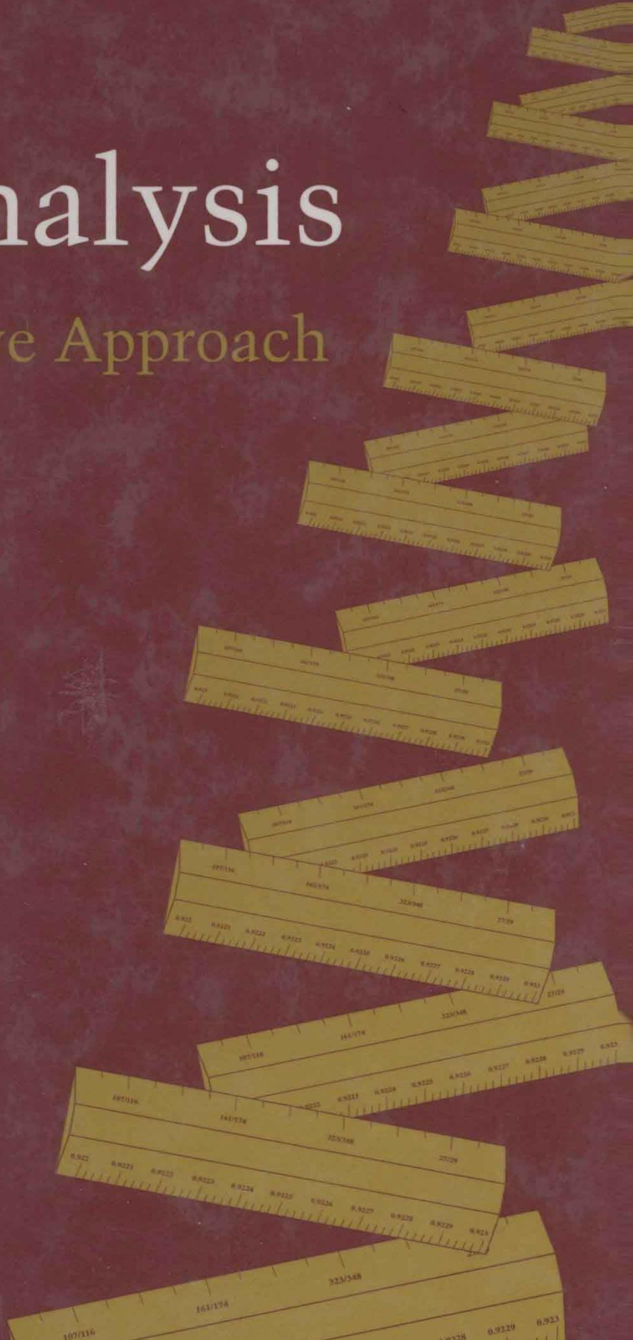
Real Analysis

A Constructive Approach



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REAL ANALYSIS

A Constructive Approach

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REAL ANALYSIS

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PREFACE

What is a constructive approach, and why should one take it?

If you look at the table of contents for this book, you'll see mostly familiar topics, but with a slightly different emphasis. There's a long chapter on the real numbers, followed by one on "An Inverse Function Theorem." The chapter on limits, sequences and series is followed by one on *uniform* continuity—why not just *pointwise* continuity? A chapter on the Riemann integral is followed by one on differentiation—but it's actually *uniform* differentiation. All of these departures from the structure of the usual real analysis text result from a careful reassessment of the role of the course in the technical education of undergraduates.

Not every student in Real Analysis is a math major, and, in many schools, only a small percentage of math majors intend to do graduate work in mathematics. A modern course is populated by a wide range of students. Some are headed for careers in secondary education, while there is often a large contingent from the physical sciences and an even larger group from computer science. These students are in the course because they need or want more than a cookbook calculus course. Some need to know more about computability and calculability of floating-point numbers, hence more about the actual nature of the reals. They also need to know about continuity because they need to know about approximations; some need to know about convergence and improper integrals because they need to know about computing special functions and transforms.

But real analysis is not primarily focused on computing. It is, significantly, a course that shapes the way students think about mathematics. Very often it is a student's introduction to precise reasoning and writing.

So. I begin with a careful construction of the real numbers, the field on which most of analysis is played. The approach here, due to Gabriel Stolzenberg, is via intervals of rational numbers and the arithmetic of such intervals. The many elementary theorems about the properties of this arithmetic later reappear as properties of the real numbers, and verifying them provides a gentle introduction to the art and practice of devising and writing readable and correct proofs. Furthermore, there is a useful metaphor: a rational interval is exactly what is obtained when a scientist uses instruments of limited (but known) accuracy to measure something. Families of rational intervals then correspond to multiple measurements, and the condition on a family that any two of its intervals must meet establishes the consistency of its measurements. Finally, a real number is defined to be a family of rational intervals that is consistent in this sense, and that contains intervals of arbitrarily small length. Interval arithmetic, carried over to families of intervals, now becomes real arithmetic, and conditions on the lengths of intervals become the properties of approximation of reals by rationals.

At this point, the students see that the reals have a far more complex structure than the rationals. One important example is the traditional Law of Trichotomy, namely that precisely one of $x < y$, $y < x$, or $x = y$ must hold. This property holds for the rationals, since rational arithmetic is basically integer arithmetic. However, reals can, in general, only be approximated by rationals. Modern computer algebra systems allow the user to specify a tolerance, which is expressed as the number of decimal places. This number can be chosen as large as one pleases, but not infinitely large. The corresponding tolerance, say ϵ, tells us how closely we can distinguish reals using the computer's rational representation. These considerations lead to the formulation of real number comparison that we prove and use throughout the book.

ϵ-Trichotomy *Given any tolerance $\epsilon > 0$, then for any reals x and y, $x < y$, $y < x$, or x and y are within ϵ of each other.*

Thus, a construction of the reals based on rational measurement and an analysis of what we can actually calculate produces a concordance of theory and practice that students of the sciences easily relate to.

Using the notion of ϵ-trichotomy as a tool for comparing real numbers enables us to describe a bisection-like algorithm for finding the inverse of a function f, providing it satisfies upper and lower bounds on its difference quotient $\frac{f(y)-f(x)}{y-x}$. This leads directly to the construction of nth root, exponential and logarithm functions.

Another hallmark of the constructivist program is its emphasis on uniform vs. pointwise continuity:

- *f is pointwise continuous at a if, given any $\epsilon > 0$ we can find a $\delta_a(\epsilon) > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta_a(\epsilon)$.*
- *f is uniformly continuous on S if, given any $\epsilon > 0$ we can find a $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta(\epsilon)$ and $x, y \in S$.*

Uniform continuity on S implies pointwise continuity at each point of S, but the converse is not true: there is no general procedure for *constructing* a single δ from the infinitely many δ_a. Not only is uniform continuity a stronger notion, it is the more desirable version of continuity since it is the one most useful in studying convergence and integrability. It turns out that the usual proofs that the basic functions of analysis are pointwise continuous also prove that these functions are uniformly continuous on appropriate intervals. We exploit this fact from the very beginning, and only use the stronger and more important uniform version of continuity.

We take a similar approach to differentiability. Instead of talking about the derivative of a function at a point, we talk about the derivative *function* on an interval. As with uniform continuity, this notion of uniform differentiability is the one that is of most importance in later theory and applications. In fact, it is an approach that generalizes readily to vector-valued functions of several variables.

An important consequence of using uniform notions is that they produce transparent proofs of important theorems such as the existence of the Riemann integral and the Fundamental Theorem of Calculus.

The pointwise versions of continuity and differentiability do lead to a number of classical examples of functions which are or aren't continuous or differentiable on various dense or nowhere dense subsets of intervals. Since we are emphasizing uniform notions, these examples are relegated to discussions in an appendix and a few exercises, which can be covered at the discretion of the instructor.

In summary, then, this is neither a text in numerical analysis nor one intended solely to prepare students to be professional mathematicians. It is a thoroughly rigorous modern account of the theoretical underpinnings of calculus; and, being constructive in nature, every proof of every result is direct and ultimately computionally verifiable (at least in principle). In particular, existence is never established by showing that the assumption of non-existence leads to a contradiction. By looking through the index or table of contents, you'll see that nothing of importance for undergraduates has been left off or compromised by our approach. The payoff of the constructive approach, however, is that it makes sense—not just to math majors, but to students from all branches of the sciences.

ACKNOWLEDGMENTS

About fifteen years ago, Gabe Stolzenberg lent me a copy of notes he had created and used to teach undergraduate seminars and directed studies courses in real analysis. I found Gabe's approach so elegant, and the material so appealing pedagogically, that I signed up to teach the analysis course, which had just become required for our majors. Over the years, with his help and encouragement, I worked this material into a text suited to the particular mix of students who take this course here at Northeastern. Many of the mathematical ideas in the current form of this text were adapted directly from Gabe's notes, especially the following: all of the material on the construction of the reals via rational interval arithmetic, the Inverse Function Theorem and its beautiful proof, the properties of exponential functions, and the creation of the Riemann integral. The use of the uniform notions of continuity and differentiability were also part of the "constructive mindset" that Gabe introduced me to, and which I have tried to employ throughout the remainder of the book. As a mathematician specializing in homological algebra, working with this new perspective was like writing a second thesis, with Gabe as advisor and friend.

Because Gabe's creative interests have taken him in other directions, this text could not be a joint authorship. He has continued to refine and expand his exposition of constructive mathematics, some of which can be found on his website. I am quite indebted to him—many of the good things in this book are due, directly or indirectly, to Gabe, and whatever is not so good is solely my responsibility.

Professor Joseph Alper read the entire manuscript and offered many extremely helpful suggestions and I am most indebted to him for this effort. Professor Robert Seeley very generously clarified a number of mathematical issues for me—especially those relating to Fourier analysis—as did my long-time office-mate Professor John Frampton. My wife, Maxine Bridger, not only proofread a lot of mathematics, she also kept me honest by countering my constructivist mindset with many a healthy platonist riposte.

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MARK BRIDGER

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INTRODUCTION (MOSTLY FOR INSTRUCTORS)

Formally, Real Analysis—the course—is a presentation of the theoretical underpinnings of calculus. It is about the Big Three: continuity, differentiability and convergence. Yet it is also, for many, an introduction to reading, writing, and thinking mathematics. I have tried to address all of these issues in this book.

In the first chapter we construct the real numbers, starting with the rationals. This lays the groundwork for the entire book. The basic concept here is that of a family of rational intervals. A real number is a family of rational intervals which satisfies two important conditions: *consistency* (any two intervals in the family intersect) and *fineness* (the family contains intervals of arbitrarily small length). These conditions, together with the arithmetic that families inherit from the rationals, lead to all of the familiar algebraic properties of the reals. We establish these properties via quite a few propositions and one main theorem (completeness). Proving these results requires

- knowing simple properties of the arithmetic of rational numbers,
- applying elementary algebra and simple logic, and
- learning to apply new definitions and newly proved results.

Thus, Chapter 1 is critical because it provides not only the mathematical ideas that permeate the rest of the text, but also the introduction to the reasoning and writing skills necessary for doing and communicating mathematics. Nothing is more boring than having to read a seemingly endless theorem-proof sequence, so I have tried to provide just enough sample proofs and hints so that readers can proceed on their own. Many propositions are left as exercises; indeed, the exercises provide a vital part of the whole pedagogical process. I take this chapter at a leisurely pace, allowing students to write, critique, and rewrite their work. It is an investment of time well worth making early in the course.

Chapter 1 ends with what may be the central result of any real analysis course: the completeness of the reals. This is expressed in terms of families of real intervals, but in Chapter 3 it is rephrased in the language of Cauchy sequences.

Chapter 2 uses the Completeness Theorem to prove the useful Inverse Function Theorem. This, in turn, is used to construct nth roots, general exponential functions, and logarithms. A section is devoted to the Euler number e and the natural logarithm.

Chapter 3 introduces sequences, limits, and series and derives basic formulas and inequalities for the various functions already constructed.

In Chapter 4 we encounter uniform continuity. Since this version of continuity is the one most used in more advanced courses, we relegate the idea of pointwise continuity to the exercises. Nothing is lost, however, since the usual verifications of pointwise continuity for the basic functions of calculus are used with little modification to establish uniform continuity of these functions on intervals. We also encounter many interesting and important consequences of uniform continuity, among them boundedness and the extension of uniformly continuous functions from dense subsets—for example, extending functions from a punctured interval $[a,b]-\{x_1,\ldots,x_n\}$ to the closed interval $[a,b]$.

In Chapter 5, we use the Completeness Theorem again, this time to construct the Riemann integral $\int_a^b f$ for functions uniformly continuous on an interval $[a,b]$. The results previously established for limits and extensions of uniformly continuous functions can now be applied to define and calculate improper integrals. It is here that we introduce the important idea of functions defined as integrals. This includes the definition of the arctangent as an integral, an alternate definition of the natural logarithm (previously defined as an inverse function), and the use of improper integrals to construct the Gamma function and Laplace transforms.

Chapter 6 on differentiation emphasizes the derivative as a function rather than a pointwise limit. All the usual formulas from calculus are derived. In particular, the uniform version of differentiability that we use makes for very short and illuminating proofs of two central results of calculus:

- **The Law of Bounded Change**, which says that bounds for the derivative (i.e. $A \le f'(x) \le B$) are bounds for the difference quotient (i.e. $A \le \dfrac{f(y)-f(x)}{y-x} \le B$). (This is sometimes called the "Mean Value Inequality.")
- **The Fundamental Theorem of Calculus**.

In this chapter, we also derive some rather more difficult results on differentiating under the integral sign. In the case of improper integrals, we introduce "dominated convergence" assumptions, which we will also use later in studying series of functions.

In Chapter 7, nearly all of the ideas developed in the course are applied to studying the properties of sequences and series of continuous and differentiable functions. The particular case of power series is given special attention. The chapter ends with the definition of the periodic (trigonometric) functions as power series and a derivation of their properties (including a definition of π)—*all without pictures.* My students invariably enjoy this; in fact, with just a few simplifications and detours, it has even worked well for high school students taking AP calculus.

The last chapter of the book is organized around Fourier series, but it also provides an introduction to some of the more advanced ideas in functional analysis: inner products of functions, the Bessel and Cauchy-Schwartz inequalities and their applications, kernels and convolutions, and Abel summability. The early sections also introduce the complex numbers and the properties of complex-valued functions of a real variable.

There is enough material in the eight chapters to give a full-year course, especially if a lot of the more challenging exercises are assigned and discussed in class. Some of the exercises which have several parts and require more extensive work are labeled "projects."

I have usually given Real Analysis as a one-semester course. I generally get to cover the following.

1. Chapter 1: sections 1.0 through 1.7 (omitting 1.8 and skimming some of the material on absolute value and betweenness).
2. Chapter 2: in which I skip the more technical results—especially the 1- and 2-sided versions of the Inverse Function Theorem and some of the inequalities relating to the Euler number e.
3. Chapter 3: just what I need to talk about convergence of series.
4. Chapter 4: section 4.1 and the beginning of section 4.2 (omitting extensions of continuous functions), some material on limits from Chapter 3.
5. Chapter 5: sections 5.1 and 5.2.
6. Chapter 6: sections 6.1 through 6.3.
7. Chapter 7: just the material on power series.

Having done this for one semester, if there is enough student interest in a second semester, or a student wants to do a reading course, I can cover the more technical topics such as improper integrals, general convergence of sequences of functions, complex numbers, and Fourier series.

After teaching Real Analysis for many years, I'd say that my general experience has been that there is no general experience. Student ability, background and motivation can vary a lot from year to year, and I think it is a mistake to commit to a strict syllabus before you know your class. What is critical is that students do lots of problems and write lots of proofs. It is also very important that the central definitions and examples be memorized. I give several quizzes devoted exclusively to this. On the other hand, the more difficult material (proofs) is best tested via problem sets. Students seem to do these best—and enjoy them more—when working with one or two others. (But I do require independent write-ups!)

In terms of submitting mathematical work, most students initially write it out by hand. Since I typically require rewrites, many soon learn to use an equation editor with their word-processor. The software package *Scientific Notebook* is a good alternative, especially if you can get your school to underwrite its purchase. I have even had a few ambitious students learn to use TEX or LATEX.

It is important to remember that this is an undergraduate course, and that most students taking it are probably not intending to go to graduate school in theoretical mathematics. The goal here is to have students understand the mathematics, be able to create some on their own, and come away with happy memories of the experience. There is also plenty of challenging material here, especially in the problems, for the talented and highly motivated student. The approach I have taken in this book has worked well over the years for me and my students. I hope it does for you and yours as well.

CONTENTS

0. PRELIMINARIES

0.1 The Natural Numbers

You have to begin *somewhere*. We begin with the whole numbers: 0, 1, 2, ...and assume that we know what they are and that they have all the basic properties we know and love. Here are some of them:

1. (Commutative laws) $m + n = n + m$, $mn = nm$
2. (Associative laws) $k + (m + n) = (k + m) + n$, $k(mn) = (km)n$
3. (Distributive law) $k \cdot (m + n) = k \cdot m + k \cdot n$
4. (Additive identity) $m + 0 = m$
5. (Multiplicative identity) $m \cdot 1 = m$
6. (Cancellation)
   (a) If $m + k = n + k$, then $m = n$.
   (b) If $m \cdot k = n \cdot k$ and $k \neq 0$, then $m = n$.
7. (Inequalities)
   (a) $m < n$ if and only if there is a non-zero whole number k with $m + k = n$.
   (b) $m \leq n$ if and only if $m > n$ is false.
   (c) For any k, $m + k < n + k$ if and only if $m < n$ (same for $\leq$).
   (d) For any $k \neq 0$, $m \cdot k < n \cdot k$ if and only if $m < n$ (same for $\leq$).
   (e) (Trichotomy) For any m, n, either $m < n$, $n < m$, or $m = n$.

There are, of course, many more such properties, but we will not attempt to list them all, nor will we try to prove any of them. Attempts have been made to derive the whole numbers and their properties solely from the "laws of logic" or from certain axioms for set theory, but we will not go down that path. In fact, it is not even clear that such an approach is worthwhile, since the existence and properties of whole numbers is arguably as basic and intuitive as the laws of logic or set theory (perhaps even more so).

We will denote the collection or set of whole numbers (including 0) by $\mathbb{N}$, standing for *natural numbers.*

NOTATION 0.1.1 *$n \in \mathbb{N}$ means that n is a natural number.*

One of the most useful properties of $\mathbb{N}$ is the following, which we have put in a box because of its importance.

Principle of Mathematical Induction
Suppose that S is a collection or set of natural numbers with the properties: (a) S contains the number 0, and (b) whenever S contains the number n it also contains $n+1$. Then S is actually all of $\mathbb{N}$.

There are several alternative and equivalent versions of this principle; the version you use depends on the nature of the result you want to prove.

Variation 1: Suppose S contains k, and whenever S contains n it also contains $n+1$; then S contains all natural numbers $\geq k$.

Variation 2: Suppose S contains 0, and whenever S contains all the numbers from 0 through n it also contains $n+1$; then $S = \mathbb{N}$.

Mathematical induction is often compared to the behavior of dominos. The dominos are stood up on edge close to each other in a long row. When one is knocked over, it hits the next one (analogous to n in S implies $n+1$ in S), which in turn hits the next, etc. If then we hit the first (0 in S), then they will all eventually fall (S is all of $\mathbb{N}$). In Variation 1 above, we start by knocking over the kth domino, so that it and all subsequent ones eventually fall.

Here is an example of how a proof by induction works.

EXAMPLE 0.1.2 *Prove that for any $n \geq 1$, the sum of the first n odd numbers is n^2.*

Proof. We use Variation 1 above with $k = 1$. We first verify the claim when $n = 1$: the first odd number is 1 and the first square is $1^2 = 1$, so the claim holds in this case. Now we make the so-called "induction assumption" (or "induction hypothesis"), namely that the claim is true for some $n \geq 1$; so we have

$$1 + 3 + 5 + \cdots + \overbrace{(2n-1)}^{n\text{th odd number}} = n^2.$$

The idea is to use this to prove that the claim is true for the next number, $n+1$. So, starting with this equation, let's add the next odd number, $2n+1$, to both sides:

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = n^2 + (2n+1).$$

The left-hand side is the sum of the first $n+1$ odd numbers, while the right-hand side is, of course, equal to $(n+1)^2$. Thus, whenever the claim is true for a natural number $n \geq 1$ it is also true for $n+1$. All the dominos starting from $n = 1$ fall, and our proof is complete. ■