Discrete Mathematics

WITH COMBINATORICS

SECOND EDITION

JAMES A. ANDERSON

Discrete Mathematics with Combinatorics

SECOND EDITION

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PEARSON
Prentice
Hall

Upper Saddle River, New Jersey 07458

Library of Congress Cataloging-in-Publication Data
Anderson, James A. (James Andrew)
Discrete mathematics with combinatorics.—2nd ed. / James A. Anderson; with the assistance of Jerome Lewis and O. Dale Saylor.
p. cm.
Includes bibliographical references and index.
ISBN: 0-13-045791-4
1. Mathematics. 2. Combinatorial analysis. I. Lewis, Jerome (Jerome L.) and Saylor, O. Dale. II. Title.
QA39.2.A534 2004
510—dc21 2003054831

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Cover Photo: *Adolph Gottlieb (1903–1974), Composition, 1955. The Museum of Modern Art/Licensed by Scala-Art Resource, NY. © VAGA, NY*
Text Art: *Kristin and Philip Muzik*

Pearson Prentice Hall
Pearson Education, Inc.
Upper Saddle River, New Jersey 07458

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

ISBN 0-13-045791-4

Pearson Education LTD., *London*
Pearson Education Australia PTY, Limited, *Sydney*
Pearson Education Singapore, Pte. Ltd
Pearson Education North Asia Ltd. *Hong Kong*
Pearson Education Canada, Ltd., *Toronto*
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SYMBOLS

LOGIC

1.	$p \wedge q$	p and q	page 2
2.	$p \vee q$	p or q	page 2
3.	$\sim p$	not p	page 2
4.	$\underline{\vee}$	exclusive or	page 4
5.	$p \rightarrow q$	if p then q	page 9
6.	$p \leftrightarrow q$	p if and only if q	pages 10 and 17
7.	$p \equiv q$	p is logically equivalent to q	page 13
8.	**T**	statement that is a tautology	page 15
9.	**F**	statement that is logically false	page 15
10.	$\therefore$	therefore	page 19
11.	$\mid$	Sheffer stroke (nand)	page 29
12.	$\downarrow$	Pierce's arrow (nor)	page 29
13.	$\forall$	for all	page 98
14.	$\exists$	There exists	page 98

SETS

15.	$\{a_1, a_2, a_3, \ldots, a_n\}$	set containing $a_1, a_2, a_3, \ldots, a_n$	page 51
16.	$\{x : P\}$	set containing all elements having property P	page 51
17.	$a \in A$	a is an element of the set A	page 52
18.	$A \subseteq B$	the set A is a subset of the set B	page 52
19.	$\varnothing$	empty set or null set	page 52
20.	U	universal set	page 52
21.	$A \cap B$	intersection of sets A and B	page 54
22.	$A \cup B$	union of sets A and B	page 55
23.	$A - B$	set difference of sets A and B	page 56
24.	$A \triangle B$	symmetric difference of sets A and B	page 56
25.	A'	complement of the set A	page 56
26.	$\mathcal{P}(A)$	power set of the set A	page 57
27.	$A \times B$	cartesian product of sets A and B	page 58
28.	(a, b)	ordered pair	page 58
29.	aRb	a is related to b by the relation R	page 72
30.	R^{-1}	inverse relation of R	page 73
31.	$S \circ R$	composition of relations S and R	page 73
32.	$(S, \leq)$	partially ordered set with ordering $\leq$	page 80
33.	$\bigcap_{i=1}^{n} A_i$	The intersection of sets $A_1, A_2, \ldots, A_n$	page 118
34.	$\bigcup_{i=1}^{n} A_i$	The union of sets $A_1, A_2, \ldots, A_n$	page 118

FUNCTION

35.	$f : A \rightarrow B$	function f from A to B	page 87
36.	$f(E)$	image of E	page 87
37.	$f^{-1}(F)$	preimage of F	page 87
38.	$f \circ g$	composition of f and g	page 88
39.	f^{-1}	inverse function	page 89
40.	$f(x) = \lfloor x \rfloor$	floor function	page 142
41.	$f(x) = \lceil x \rceil$	ceiling function	page 142
42.	$f(x) = x!$	factorial function	page 142

43. $O(g(n))$	big-Oh of $g(n)$	page 182
44. δ_{ij}	Kronecker δ	pages 215 and 733

■ CIRCUIT DIAGRAMS

45.	and gate	page 39
46.	or gate	page 39
47.	not gate	page 39
48.	nand gate	page 40
49.	nor gate	page 40
50. $p \cdot q$	Boolean multiplication (p and q)	page 38
51. $p + q$	Boolean addition (p or q)	page 38
52. p'	Boolean complement (not p)	page 38

■ GRAPHS AND TREES

53. $G(V, E)$	graph with vertices V and edges E	pages 75 and 227
54. $\{a, b\}$	edge connecting vertices a and b	pages 75 and 227
55. (a, b)	directed edge from vertex a to vertex b	page 233
56. $deg(v)$	degree of a vertex v	page 229
57. K_n	complete graph with n vertices	page 231
58. $K_{m,n}$	complete bipartite graph with m and n vertices	page 231
59. $v_0 v_1 v_2 \cdots v_n$	path of length n from v_0 to v_n	page 230
60. $G^c(V, E')$	complement of graph $G(V, E)$	page 519
61. $C_G(\lambda)$	number of ways of coloring graph $G(V, E)$ using λ colors	page 536
62. $G_{\hat{e}}$	graph G with edge e removed	page 536
63. $G_{/e}$	graph G with edge $e = \{a, b\}$ removed and vertices a and b identified	page 536
64. $cl(G)$	closure of graph $G(V, E)$	page 547
65. $c(e)$	capacity of an edge e	page 620
66. $f(e)$	flow in an edge e	page 621
67. $val(f)$	value of a flow f	page 623
68. (S, T)	cut of a network into S and T	page 623
69. $C(S, T)$	capacity of a cut (S, T)	page 623

■ NUMBER THEORY

70. Z	set of integers	page 105
71. N or Z^+	set of positive integers	page 107
72. $a \mid b$	a divides b	page 119
73. $\gcd(a, b)$	greatest common divisor of a and b	page 121
74. $\text{lcm}(a, b)$	least common multiple	page 122
75. $a \equiv b \pmod{n}$	a congruent to b modulo n	page 129
76. $[a]$	equivalence class containing a	page 84
77. $[a] \oplus [b]$	addition of congruence classes	page 131
78. $[a] \odot [b]$	multiplication of congruence classes	page 131
79. $[[m]]_n$	smallest positve integer congruent to m modulo n	page 131
80. ϕ	Euler ϕ function	page 401
81. $\text{ord}_n\, a$	order of a modulo n	page 406

■ MATRICES

82. $[A_{ij}]$	matrix with entry A_{ij}	page 146
83. $A + B$	matrix sum	page 148
84. aA	matrix scalar product	page 148

85. AB	matrix product	page 149
86. A^t	transpose of matrix A	page 151
87. $A \odot B$	Boolean product of matrices	page 152
88. $\overline{A_{ij}}$	matrix obtained by removing ith row and jth column from A	page 213
89. $\|A\| = \det(A)$	determinant of the matrix A	page 214
90. I_n	$n \times n$ identity matrix	page 215
91. A^{-1}	multiplicative inverse of the matrix A	page 216
92. $Adj(A)$	adjoint matrix of A	page 216

■ RECURSION AND COUNTING

93. $n!$	n factorial	page 142
94. $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$	summation notation	page 144
95. $P(n, r) = \frac{n!}{(n-r)!}$	number of r permutations on n objects	page 303
96. $C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$	number of r combinations on n objects	page 305
97. $P(n : n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!n_2!\ldots n_k!}$	generalized permutation	page 336
98. $C(n : n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!n_2!\ldots n_k!}$	generalized combination	page 324
99. $P(A)$	probability of A	page 317
100. $P(B \mid A)$	probability of B given A	page 335
101. $E(R)$	expected value of random variable R	page 340
102. μ	mean of random variable R	page 343
103. σ^2	variance of random variable R	page 343
104. $R(p, q)$	Ramsey number	page 332
105. $\text{Fib}(n)$	nth Fibonacci number	page 173
106. $\text{Cat}(n) = C_n$	nth Catalan number	page 173
107. $G(C)$	the stabilizer of coloring C	page 710
108. $\Delta^k f(x)$	kth difference of $f(x)$	page 434
109. E	operator defined by $E(f(x) = f(x+1)$	page 434
110. $x^{(n)}$	$x(x-1)(x-2)\cdots(x-n+1)$	page 437
111. $a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_2 x^{(2)} + a_1 x + a_0$	factorial polynomial	page 438
112. $x^{(-m)}$	$\frac{1}{(x+m)^m}$	page 443
113. $\binom{x}{n}$	$\frac{x^{(n)}}{n!}$	page 444
114. $s_k^{(n)}$	Stirling number of the first kind	page 444
115. $S_k^{(n)}$	Stirling number of the second kind	page 444
116. $\sum$	summation operator	page 447
117. L_n	nth lucas number	page 425
118. D_n	the number of derangements of n distinct ordered symbols	page 466
119. $r_k(c)$	the number of ways of placing k rooks on board C in nonattacking position	page 470
120. $R(x, C)$	rook polynomial on board C	page 470
121. $[t_0 : t_1, t_2, \ldots, t_n]$	continued fraction	page 281
122. $T_k^{(n)}$	the number of ways of placing n distinguishable objects in k distinguishable boxes with no box empty	page 510

■ ALGEBRA

123. $(A, \vee)$ or $(A, +)$	upper semilattice A	page 358
124. $(A, \wedge)$ or $(A, \cdot)$	lower semilattice A	page 358
125. $(S, \vee, \wedge)$	lattice S	page 365
126. $a \circ H$	left coset	page 375
127. $G_1 \oplus G_2$	direct sum of groups G_1 and G_2	page 752

128. $\langle a \rangle$	principal ideal generated by a	page 724
129. $A[x]$	set of polynomials with coefficients in A	page 736
130. χ	group character	page 751
131. $a_1a_2a_3 \cdots a_n$	string of symbols of an alphabet A	page 648
132. A^*	set of all strings of an alphabet A	page 648
133. $M = (A, S, s_0, T, F)$	automaton with alphabet A, set of states S, starting state s_0, set of acceptance states T and transition function F	page 652
134. $M(L)$	language accepted by automaton M	page 652
135. $\Gamma = (N, T, S, P)$	grammar with set of nonterminal symbols N, set of terminal symbols T, start symbol S, and set of productions P	page 673

■ CODES

136. $wt(c)$	number of ones in a string of code c	page 693
137. $C^{\perp}$	dual code of C	page 695
138. G_H	Hamming generating matrix	page 700
139. $G_H^{\perp}$	Hamming parity check matrix	page 700
140. (c, c')	Hamming distance between strings of codes c and c'	page 701
141. $D(C)$	smallest distance between any two elements of C	page 701

Discrete Mathematics with Combinatorics

SECOND EDITION

In Memory of
Elwood W. Stone and Orville H. Wiebe

To My Family
Marilyn, Andy, Kristin, and Phil
and to
Tom Head, Naoki Kimura,
Leonard S. Laws, and Edward Lee Dubowsky,
Teachers, mentors, and friends

PREFACE

As in the first edition, the purpose of this book is to present an extensive range and depth of topics in discrete mathematics and also work in a theme on how to do proofs. Proofs are introduced in the first chapter and continue throughout the book. Most students taking discrete mathematics are mathematics and computer science majors. Although the necessity of learning to do proofs is obvious for mathematics majors, it is also critical for computer science students to think logically. Essentially, a logical bug-free computer program is equivalent to a logical proof. Also, it is assumed in this book that it is easier to use (or at least not misuse) an application if one understands why it works. With few exceptions, the book is self-contained. Concepts are developed mathematically before they are seen in an applied context.

Additions and alterations in the second edition:

- More coverage of proofs, especially in Chapter 1.
- Added computer science applications, such as a greedy algorithm for coloring the nodes of a graph, a recursive algorithm for counting the number of nodes on a binary search tree, or an efficient algorithm for computing $a^b \bmod n$ for very large values of a, b, and n.
- An extensive increase in the number of problems in the first eight chapters.
- More problems are included that involve proofs.
- Additional material is included on matrices
- Inclusion of finite states with output and Turing machines.
- True-False questions at the end of each chapter.
- Summary questions at the end of each chapter.
- A glossary at the end of each chapter.
- Functions and sequences are introduced earlier (in Chapter 2).

Calculus is not required for any of the material in this book. College algebra is adequate for the basic chapters. However, although this book is self-contained, some of the remaining chapters require more mathematical maturity than do the basic chapters, so calculus is recommended more for giving maturity, than for any direct uses.

This book is intended for either a one- or two-term course in discrete mathematics. The first eight chapters of this book provide a foundation in discrete mathematics and would be appropriate for a first-level course for freshmen or sophomores. These chapters are essentially independent, so that the instructor can pick the material he/she wishes to cover. The remainder of the book contains appropriate material for a second course in discrete mathematics. These chapters expand concepts introduced earlier and introduce numerous advanced topics. Topics are explored from different points of view to show how they may be used in different settings. The range of topics include:

Logic—Including truth tables, propositional logic, predicate calculus, circuits, induction, and proofs.

Set Theory—Including cardinality of sets, relations, partially ordered sets, congruence relations, graphs, directed graphs, and functions.

Algorithms—Including complexity of algorithms, search and sort algorithms, the Euclidean algorithm, Huffman's algorithm, Prim's algorithms, Warshall's algorithm, the Ford-Fulkerson algorithm, the Floyd-Warshall algorithm, and Dijkstra's algorithms.

Graph Theory—Including directed graphs, Euler cycles and paths, Hamiltonian cycles and paths, planar graphs, and weighted graphs.

Trees—including binary search trees, weighted trees, tree transversal, Huffman's codes, and spanning trees.

Combinatorics—including permutations, combinations, inclusion-exclusion, partitions, generating functions, Catalan numbers, Sterling numbers, Rook Polynomials, derangements, and enumeration of colors.

Algebra—Including semigroups, groups, lattices, semilattices, Boolean algebras, rings, fields, integral domains, polynomials, and matrices.

There is extensive number theory and algebra in this book. I feel that this is a strength of this book, but realize that others may not want to cover these subjects. The chapters in these areas are completely independent of the remainder of the book and can be covered, or not, as the instructor desires. This book also contains probability, finite differences, and other topics not usually found in a discrete mathematics text.

■ Organization

The first three chapters cover logic and set theory. It is assumed in this book that an understanding of proofs is necessary for the logical construction of advanced computer programs.

The basic concepts of a proof are given and illustrated with numerous examples. In Chapter 2, the student is given the opportunity to prove some elementary concepts of set theory. In Chapter 3, the concept of an axiom system for number theory is introduced. The student is given the opportunity to prove theorems in a familiar environment. Proofs using induction are also introduced in this chapter. Throughout the remainder of the book, many proofs are presented and many of the problems are devoted to proofs. Problems, including proofs, begin at the elementary level and advance in level of difficulty throughout the book.

Relations, functions, and graphs are introduced in Chapter 2. Functions are then continued in Chapter 4. However, the development of functions in Chapter 4 is independent of the material in Chapter 2. Similarly, the development of graphs in Chapter 6 does not depend on their development as relations in Chapter 2.

Matrices, permutations, and sequences are introduced in Chapter 4 as special types of functions. Further properties of functions and matrices follow in Chapter 5. Algorithms for matrices are introduced and further properties of matrices are developed, which will be used in later chapters on algebra, counting, and theory of codes.

Permutations are used for counting in Chapter 8 and also for applications in algebra and combinatorics in later chapters. Again, the material in Chapter 8, while related to Chapter 4, can be studied independently.

Chapter 5 is independent of the previous chapters except for the matrices in the previous chapter. Algorithms are developed, including sorting algorithms. The complexity of algorithms is also developed in this chapter. Prefix and suffix notation are introduced here. They are, again, discussed in Chapter 15 with regard to traversing binary trees. Binary and hexadecimal numbers are also introduced in Chapter 5.

Many elementary concepts of graphs, directed graphs, and trees are covered in Chapter 6. These concepts are covered in more depth in Chapters 14–16. Chapter 6 is independent of the previous chapters.

In Chapters 7 and 10, the basics of number theory are developed. These chapters are necessary for applications of number theory in Chapter 22, but are otherwise completely independent of the other chapters and may be omitted if desired.

Chapter 8 is the beginning of extensive coverage of combinatorics. This is continued in many of the chapters including Chapters 12, 13, and 17. Chapter 8 also introduces basic ideas in probability which is not common in most other discrete mathematics books.

Chapters 9 and 20 cover the basic concepts of algebra, including semigroups, groups, semilattices, lattices, rings, integral domains, and fields. These chapters use Sections 3.6, and 4.3 for examples of groups and rings. Chapter 9 is necessary for the applications in Chapters 17–21.

In many ways Chapters 11, 12, and 13 form a cluster. Recursion is continued in Chapter 11. In addition to the standard linear recurrence relations normally covered in a discrete mathematics text, the theory of finite difference is also covered. Chapter 6 should be covered before this chapter unless the student already has some knowledge of recursion. Chapter 12 continues the counting introduced in Chapter 8. It covers topics introduced in Chapter 8, such as occupancy problems and inclusion-exclusion. It also introduces derangements and rook polynomials. It is closely related to Chapter 11. Many of the same topics are covered from different points of view. One example of this is Stirling numbers. However neither chapter is dependent on the other.

Chapters 11 and 12 are tied together in Chapter 13, where generating functions are used to continue the material in both chapters. In particular, generating functions provide a powerful tool for the solution of occupancy problems.

Chapters 14–16 continue the study of trees and graphs begun in Chapter 6. They obviously depend on the material in Chapter 6, but are virtually independent of most of the preceding chapters. One exception is the use of matrices in some of the algorithms. Many of the standard topics of graphs and trees are covered, including planar graphs, Hamiltonian cycles, binary trees, spanning trees, minimal spanning trees, weighted trees, shortest path algorithms, and network flows.

Chapters 17–22 form another cluster consisting of number theory, algebra, combinatorics, and their application. The theory of computation is introduced in Chapter 17. This includes codes, regular languages, automata, grammars, Turing machines, and their relationship. This chapter uses semigroups from Section 9.2. Chapter 18 introduces special codes, such as error detecting codes and error correcting codes. This chapter requires knowledge of group theory, found in Section 9.4, and some knowledge of matrices, found in Chapters 4 and 5. Codes are explored from yet another direction in Chapter 22 where cryptography is introduced. This chapter is dependent on the previous chapters on number theory.

In Chapter 19, algebra and combinatorices are combined for the development of Burnside's Theorem and Polya's Theorem for the enumeration of colors. It primarily depends on a knowledge of permutations found in Section 9.4.

Chapter 21 is a simple application of groups and semigroups and their mapping onto the complex plane. The prerequisites for this chapter are Sections 9.2 and 9.4.

Chapter 22 gives three important applications of number theory. The study of Hashing functions and cryptography are particularly relevant to computer science.

When teaching a beginning course, I normally cover Chapters 1–5 in their entirety, Sections 8.1–8.5, and the first three sections Chapter 6. As mentioned previously, the material in the first eight chapters is arranged for maximal flexibility. The following chart shows the required prerequisites for each chapter.

Chapter	**Prequisite Chapters or Sections**
Chapter 1	None
Chapter 2	None
Chapter 3	Sections 1.1–1.4 and 2.1
Chapter 4	None
Chapter 5	Sections 4.1–4.3
Chapter 6	None
Chapter 7	Chapter 3
Chapter 8	None
Chapter 9	Sections 2.6, 2.7, and 3.6
Chapter 10	Chapter 7
Chapter 11	Sections 5.1–5.3
Chapter 12	Chapter 8
Chapter 13	Chapters 11 and 12
Chapter 14	Chapter 6
Chapter 15	Chapter 6
Chapter 17	Chapter 9
Chapter 18	Chapters 5 and 9
Chapter 19	Chapter 9
Chapter 20	Chapter 9
Chapter 21	Chapter 9
Chapter 22	Chapter 10

■ Supplements

A solutions manual is available from the publisher with complete solutions to all problems. A website is available at www.prenhall.com/janderson. This website includes links to other interesting sites in discrete mathematics, quizzes, and supplementary problems. In addition, there are two problems oriented paperbacks that can be used with the textbook: Practice Problems in Discrete Mathematics (407 pp.) by B. Obrenic and Discrete Mathematics Workbook (316 pp.) by J. Bush. The first consists entirely of problems with answers/solutions. The second contains an outline of subject, sample worked out problems, and problem sets (with answers). Each of these two supplements is free when shrinkwrapped with the text. As stand-alone items, they have prices. So the order ISBN for the textbook plus the free Obrenic supplement is 013-117279-4. The order ISBN for the textbook plus the free Bush supplement is 013-117278-6.

■ Acknowledgments

First, I would like to thank Dale Saylor for the tremendous amount of work he has contributed to the book. Without him, this book would probably not have been revised. I also want to especially thank Jerome Lewis for the computer science applications which he has contributed to the took. I am very grateful to Douglas Shier for the use of his collection of methods for counting spanning trees. I would like to thank Kristin and Philip Muzik for their excellent artwork. I am especially grateful to James Bell for the tremendous amount of work that he contributed to the first edition. I would also like to thank my colleagues Rick Chow, Dale Saylor, Debabrata Mukherjee, and Jerome Lewis for their help in checking answers to the exercises. I would like to thank Alex Osipovk, Ole-Kristian S. Losvik, Timothy Haven, and others who have sent me corrections to the first edition of the book. I would also like to thank George Lobell and Jennifer Brady at Prentice Hall and Bob Walters of Prepress Management, Inc., for their help with the book.

With apologies for the delay, I would like to thank the following reviewers of the first edition:

Anthony B. Evans, Wright State University
Madeleine Schep, Columbia College
Akihiro Kanamori, Boston University
Krishnaiya Thulasiraman, University of Oklahoma
Gabor Sarkozy, Worcester Polytechnic Institute
Alvin Swimmer, Arizona State University

I would also like to thank the following reviews of this edition:

Beth Novick, Clemson University
Thomas Hughes, Vanier College/Concordia University
John Konvalina, University of Nebraska-Omaha
George M. Butler, Louisiana Tech University
Michael Neubauer, California State University, Northridge
Myron Hlynka, University of Windsor
William G. Brown, McGill University
Ted Wilcox, Rochester Institute of Technology

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