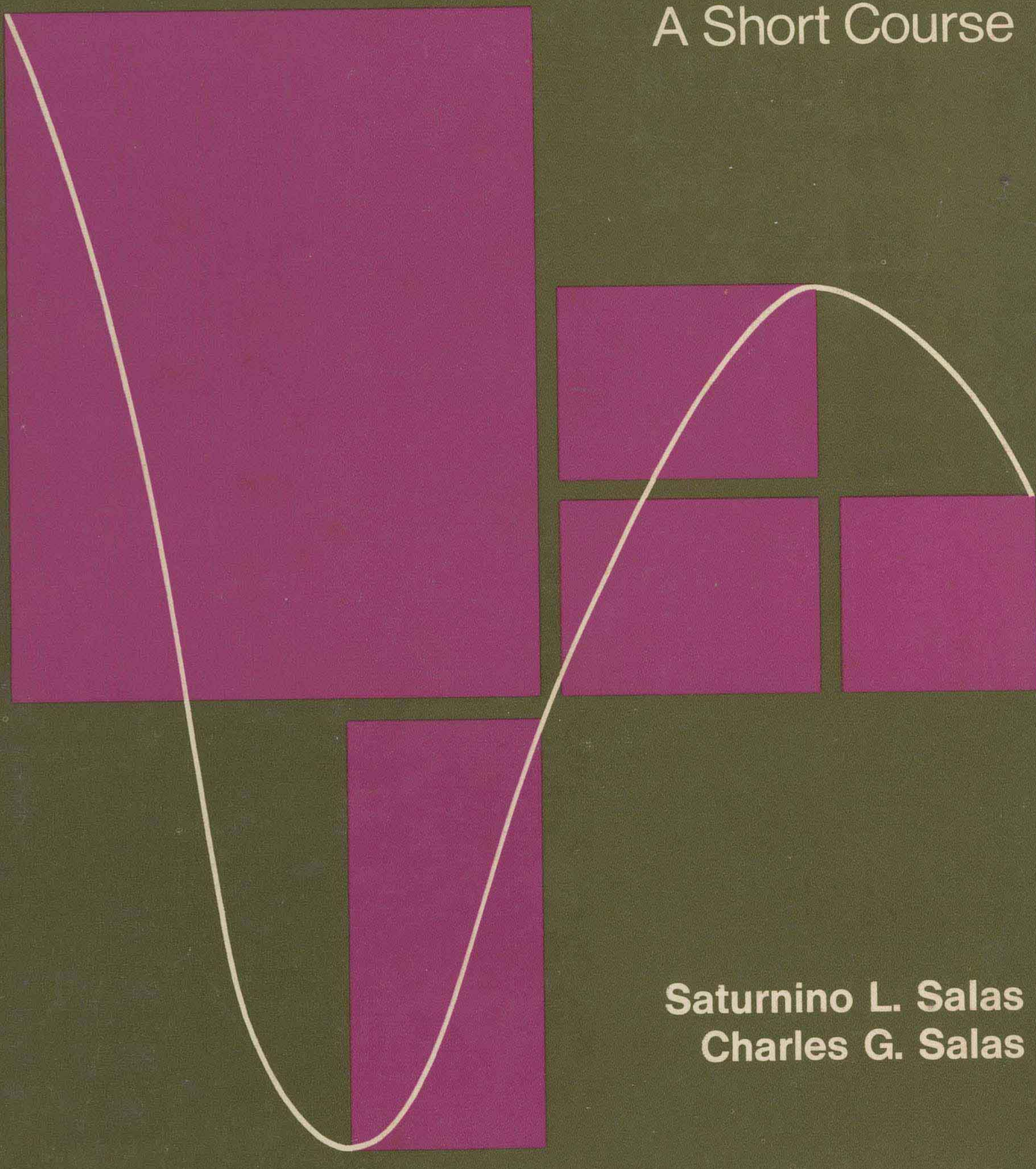
Precalculus

A Short Course



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Precalculus

To

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Preface

Precalculus: A Short Course is a review of those topics in high school mathematics that are most useful for the study of calculus. There are five chapters: the first deals with numbers and algebraic expressions, the second with analytic geometry, the third with functions, the fourth with trigonometry, and the last with such special topics as induction, the least upper bound axiom, and polar coordinates.

Designed for a semester or an intensive quarter, this book is addressed to those students who wish to study calculus but are not comfortable with their high school mathematics.

Some of the precalculus books we have seen seem more difficult than the calculus they were designed to precede. By comparison, this little book will seem easy. We have stayed with the material that we consider most useful for calculus and avoided all unnecessary sophistication.

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S.L.S. C.G.S.

The Greek Alphabet

Α	α	alpha
Β	β	beta
Γ	γ	gamma
Δ	δ	delta
Ε	ε	epsilon
Ζ	ζ	zeta
Η	η	eta
Θ	θ	theta
Ι	ι	iota
Κ	κ	kappa
Λ	λ	lambda
Μ	μ	mu
Ν	ν	nu
Ξ	ξ	xi
Ο	ο	omicron
Π	π	pi
Ρ	ρ	rho
Σ	σ	sigma
Τ	τ	tau
Υ	υ	upsilon
Φ	φ	phi
Χ	χ	chi
Ψ	ψ	psi
Ω	ω	omega

Contents

1 Numbers and Algebraic Expressions

In this chapter we focus on those parts of high school algebra that are necessary for calculus.

1.1 Classification of Numbers

We begin with the set of *natural numbers*

$$\{1, 2, 3, \ldots\}.$$

With these numbers we can count, add, and multiply, but we cannot always subtract (for example, $5 - 8$ is not a natural number) and we cannot always divide ($5 \div 8$ is not a natural number).

To be able to subtract arbitrarily, we enlarge the number system to include the set of all *integers*

$$\{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

If we wish to divide, we must allow for fractions and use the set of *rational numbers*; a rational number, you will recall, is a number of the form

$$p/q, \qquad \text{where } p \text{ and } q \text{ are integers and } q \neq 0.$$

The rational numbers include the integers ($n = n/1$) and the common fractions, both *positive* and *negative*, both *proper*† and *improper*.‡ They also include the *mixed numbers*, since these can be written as improper fractions:

$$2\tfrac{1}{3} = \tfrac{7}{3}, \qquad 4\tfrac{2}{5} = \tfrac{22}{5}, \text{ etc.}$$

† A proper fraction is a fraction with the numerator less than the denominator, e.g., $\frac{2}{3}$.
‡ An improper fraction is a fraction with the numerator greater than the denominator, e.g., $\frac{3}{2}$.

With the rational numbers we can count, add and subtract, multiply and divide, but we cannot always take roots and we cannot measure all distances. Consider, for example, a unit square. (Figure 1.1.1) By the Pythagorean theorem,† the distance between the opposite vertices of this square must be $\sqrt{2}$, the square root of 2. As we will show, this is not a rational number. With only rational numbers at our disposal we could not measure this distance. Nor could we find $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{13}$ (the cube root of 13), or $\sqrt[5]{5}$ (the fifth root of 5). The circle would also give us trouble. You are familiar with the formula

$$\text{circumference} = 2\pi r.$$

According to this formula, the distance around a unit circle ($r = 1$) would be 2π, but π is not rational and neither is 2π.‡ With only rational numbers, we could not even measure the distance around a wheel of radius 1.

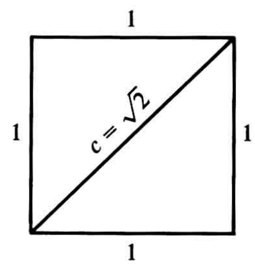

FIGURE 1.1.1

To be able to take roots of all positive numbers and to be able to measure all distances, we must enlarge the number system once again. We must accept not only the *rational* numbers but also the *irrational* numbers. Together these comprise the set of *real numbers*.§

† The Pythagorean theorem states that in a right triangle, $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse. (Figure 1.1.2) In the case of the unit square, $1^2 + 1^2 = c^2$, so that $c^2 = 2$ and $c = \sqrt{2}$.

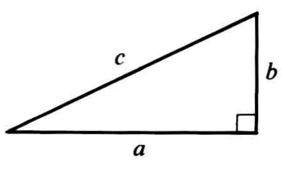

FIGURE 1.1.2

‡ That π is irrational is not so easy to prove. We leave to you the proof that 2π is irrational since π is. (Exercise 5)

§ To take even roots of negative numbers we must expand the number system once more, this time to include the *complex numbers*. This is done in Section 5.8.

Real Numbers and the Number Line

You can visualize the real number system as stretched out over a horizontal line, as in Figure 1.1.3. Choose a point for 0 and a unit of length. Then display the number r that many units to the right of 0 if r is positive, and $-r$ units to the left of 0 if r is negative. The resulting pattern is the familar *number line*, also called the *coordinate line*. Each point on it corresponds to a unique real number, and each real number corresponds to a unique point.

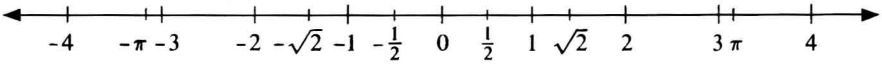

FIGURE 1.1.3

Problem. Show that $\sqrt{2}$ is not rational.

SOLUTION. If, on the contrary, $\sqrt{2}$ is rational, then we can write it as a fraction and reduce the fraction to lowest terms. Suppose then that

(*) $\sqrt{2} = p/q$ with p/q in lowest terms.

Squaring both sides, we have

$$2 = p^2/q^2,$$

and therefore

(**) $$2q^2 = p^2.$$

This is an equation between positive integers. Since p^2 is a multiple of 2, p^2 must be even, which means that p itself must be even. (Can you see why?) Thus we can write

$$p = 2r.$$

Substituting $2r$ for p in (**), we get

$$2q^2 = (2r)^2 = 4r^2$$

and, dividing by 2,

$$q^2 = 2r^2.$$

This last equation tells us that q^2 is even, and therefore that q is even.

With p and q both even, p and q are both divisible by 2, and the fraction p/q cannot be in lowest terms. But this is impossible, for we know by (*) that p/q *is* in lowest terms. Thus the assumption that $\sqrt{2}$ is rational has led to a contradiction. It follows that $\sqrt{2}$ is not rational. □

Exercises†

1. Draw a number line and mark on it the points 0, $\frac{1}{2}$, -1, 1, 2, $\frac{7}{3}$, 3.3, and -4.
2. Draw a number line and mark on it the points 0, 10, 50, -65, and -100.
3. Show that each of the following numbers is rational by expressing it as the quotient of two integers:
   *(a) $6\frac{4}{5}$. (b) $-2\frac{1}{4}$. *(c) 3.176. (d) -2.1115.
4. *Every repeating decimal represents a rational number.* Write each of the following as the quotient of two integers:
   *(a) 0.333 . . . . (b) 0.212121 . . . .
   *(c) 0.326326326 . . . . *(d) $0.a_1a_1a_1 \ldots.$
   (e) $0.a_1a_2a_1a_2 \ldots.$ *(f) $0.a_1a_2a_3a_1a_2a_3a_1a_2a_3 \ldots.$
5. Show that since π is irrational, 2π is irrational.
*6. Given only a unit length, use right triangles and the Pythagorean theorem to construct line segments of lengths $\sqrt{3}$ and $\sqrt{5}$.
7. Show that if a and b are rational, then $a + b$, $a - b$, and ab are rational, and if $b \neq 0$, then a/b is also rational.
8. (*Optional*) Show that $\sqrt{3}$ is not rational.

1.2 Some Properties of the Real Number System

For easy reference and review, we briefly go over some of the basic arithmetic properties of real numbers. Undoubtedly you are already familiar with them.

1. Addition and multiplication are *associative*:

$$\boxed{a + (b + c) = (a + b) + c, \qquad a(bc) = (ab)c}$$

and *commutative*:

$$\boxed{a + b = b + a, \qquad ab = ba.}$$

2. Multiplication *distributes* over addition:

$$\boxed{a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca.}$$

We will return to this in Section 1.6, for this is the basis of factoring.

3. The number 0 is an *additive identity*:

$$\boxed{a + 0 = 0 + a = a}$$

† The starred exercises have answers at the back of the book.

and the number 1 is a *multiplicative identity*:

$$\boxed{a \cdot 1 = 1 \cdot a = a.}$$

4. The number 0 times any number is 0:

$$\boxed{0 \cdot a = a \cdot 0 = 0.}$$

5. If the product of two numbers is 0, then at least one of the factors is 0:

$$\boxed{\text{if} \quad ab = 0, \qquad \text{then} \quad a = 0 \quad \text{or} \quad b = 0.}$$

We use this property often when solving equations.

6. Division by 0 is undefined:

$$\boxed{\text{there is no number } a/0.}$$

In particular, 0/0 is not defined.

7. The product of positive factors is positive. The product of an even number of negative factors is positive:

$$(-2)(-3) = 6, \qquad (-1)(-4)(-2)(-3) = (4)(6) = 24;$$

the product of an odd number of negative factors is negative:

$$(-2)(3) = -6, \qquad (-1)(4)(-2)(-3) = (-4)(6) = -24.$$

1.3 Addition and Subtraction of Algebraic Expressions

In adding or subtracting algebraic expressions, we combine like terms. Thus,

$$(2x - 5y + 5) + (6x + 3y + 1) = 8x - 2y + 6,$$

$$(2x - 5y + 5) - (6x + 3y + 1) = -4x - 8y + 4, \quad \text{etc.}$$

Sometimes it is useful to first reorder the terms:

$$\begin{aligned}(x^2 + 1 + x) + (3x + 5x^2 + 1) &= (x^2 + x + 1) + (5x^2 + 3x + 1)\\ &= 6x^2 + 4x + 2.\end{aligned}$$

Problem. Simplify

$$x^2 + 1 - [x^2 - x - (2x + 5)].$$

SOLUTION. Here there are two ways in which we can proceed. We can first remove the parentheses and write

$$\begin{aligned} x^2 + 1 - [x^2 - x - (2x + 5)] &= x^2 + 1 - [x^2 - x - 2x - 5] \\ &= x^2 + 1 - [x^2 - 3x - 5] \\ &= x^2 + 1 - x^2 + 3x + 5 \\ &= 3x + 6. \end{aligned}$$

Or, we can first remove the brackets:

$$\begin{aligned} x^2 + 1 - [x^2 - x - (2x + 5)] &= x^2 + 1 - x^2 + x + (2x + 5) \\ &= x^2 + 1 - x^2 + x + 2x + 5 \\ &= 3x + 6. \end{aligned}$$

The result, of course, is the same. □

Problem. Simplify

$$\tfrac{1}{2}(4x^2 + 6) - 5(1 - x^2).$$

SOLUTION. By the distributive law,

$$\tfrac{1}{2}(4x^2 + 6) = 2x^2 + 3 \quad \text{and} \quad 5(1 - x^2) = 5 - 5x^2,$$

so that

$$\begin{aligned} \tfrac{1}{2}(4x^2 + 6) - 5(1 - x^2) &= (2x^2 + 3) - (5 - 5x^2) \\ &= 7x^2 - 2. \quad \square \end{aligned}$$

Problem. Simplify

$$2x^2 + 3[4(x^2 + x + 1) - 2(x + 1)].$$

SOLUTION. First we distribute the 3 and remove the brackets:

$$2x^2 + 12(x^2 + x + 1) - 6(x + 1).$$

Then we distribute the 12 and the 6:

$$2x^2 + 12x^2 + 12x + 12 - 6x - 6.$$

Finally we collect like terms:

$$14x^2 + 6x + 6. \quad \square$$

Problem. Evaluate

$$3(2x - y) + 2(4x + y - 1) \qquad \text{at } x = 1, \quad y = 3.$$

SOLUTION. We substitute 1 for x and 3 for y:

$$\begin{aligned} 3[(2)(1) - 3] + 2[(4)(1) + 3 - 1] &= 3(2 - 3) + 2(4 + 3 - 1) \\ &= (3)(-1) + (2)(6) \\ &= -3 + 12 \\ &= 9. \quad \square \end{aligned}$$

Exercises

Carry out the indicated operations and simplify:

*1. $4(x - 4)$.

2. $3a(x - 1)$.

*3. $14(x - y + 2)$.

*4. $\frac{1}{4}b(x - 4)$.

5. $2[x - (y + 1)]$.

*6. $2[3(y - a) + 4a]$.

*7. $\frac{2}{33}(33y - 11x)$.

8. $\frac{1}{19}(19y - 38x^2)$.

*9. $\frac{3}{5}(15a^2 - 20x^2)$.

*10. $x^2 - 1 - [x^2 - x - (2x + 5)]$.

11. $2(2x^2 - 1) + (x^2 + x + 3)$.

*12. $212[(y^2 + 1) - (x^4 + 1) - (y^2 - x^4)]$.

13. $\frac{1}{4}x + 4[x + (4 - 2x)]$.

*14. $\frac{1}{2}(4abc - 1) - \frac{1}{3}(15abc + 6)$.

15. $4(xy + ba) - \frac{1}{2}(yx - ab)$.

*16. $2[(y^2 - x^2) + 4(x^2 - y^2) - y(1 - x)]$.

17. $6[\frac{1}{2}(a - b) - \frac{2}{3}(a + b)]$.

*18. $\frac{3}{5}[10a^2x - \frac{5}{3}a^2(x - 1) + 10a(a - 1)]$.

19. $3[(a - b)x - 2(a + b)x]$.

Evaluate:

20. $6(x - y + 2)$ at $x = 3, y = 4$.

*21. $2(3x - y) - (4x + y)$ at $x = 0, y = 1$.

22. $4(xyz - 1) - 2(3 - xyz)$ at $x = 1, y = 2, z = -2$.

*23. $10(2x + 1) + 100(x - 5) - (3x - 2)$ at $x = 4$.

24. $a(ax^2 - y) - 2a(y + ax^2) - a(x^2 + y^2)$ at $x = -1, y = 1$.

1.4 Multiplication of Algebraic Expressions

The basic ideas here are the laws of exponents and the distributive law.

Integral Powers

If a is a real number and n is a positive integer, then the *nth power* of a is defined by setting

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}.$$

Here a is called the *base* and n the *exponent*.