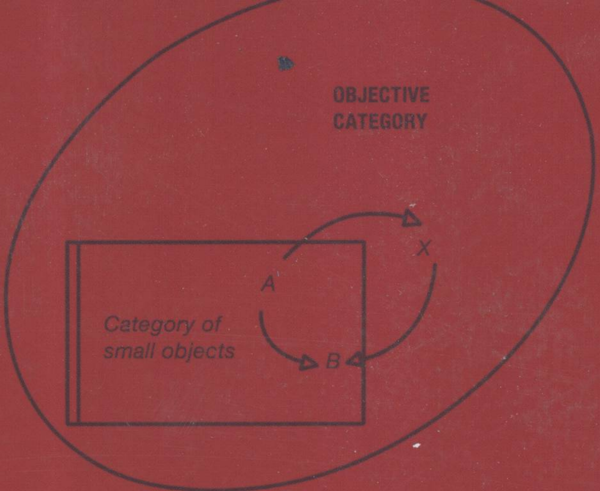

Conceptual Mathematics

A first introduction to categories

Second Edition

F. William Lawvere
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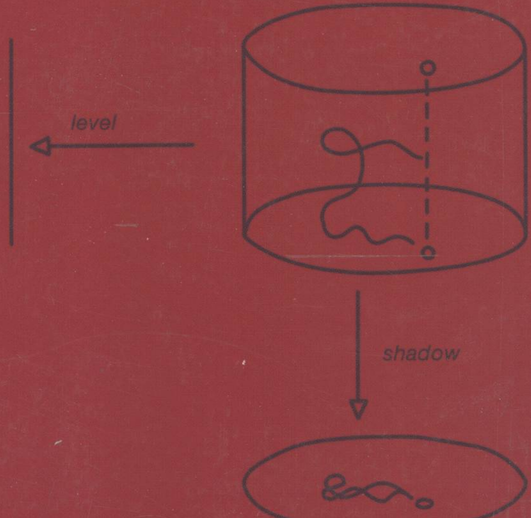

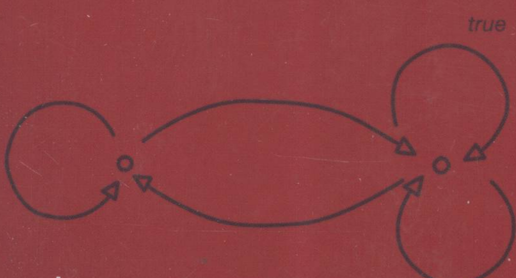

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Conceptual Mathematics, 2nd Edition

A first introduction to categories

F. WILLIAM LAWVERE
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to Fatima

Preface

Since its first introduction over 60 years ago, the concept of category has been increasingly employed in all branches of mathematics, especially in studies where the relationship between different branches is of importance. The categorical ideas arose originally from the study of a relationship between geometry and algebra; the fundamental simplicity of these ideas soon made possible their broader application.

The categorical concepts are latent in elementary mathematics; making them more explicit helps us to go beyond elementary algebra into more advanced mathematical sciences. Before the appearance of the first edition of this book, their simplicity was accessible only through graduate-level textbooks, because the available examples involved topics such as modules and topological spaces.

Our solution to that dilemma was to develop from the basics the concepts of directed graph and of discrete dynamical system, which are mathematical structures of wide importance that are nevertheless accessible to any interested high-school student. As the book progresses, the relationships between those structures exemplify the elementary ideas of category. Rather remarkably, even some detailed features of graphs and of discrete dynamical systems turn out to be shared by other categories that are more continuous, e.g. those whose maps are described by partial differential equations.

Many readers of the first edition have expressed their wish for more detailed indication of the links between the elementary categorical material and more advanced applications. This second edition addresses that request by providing two new articles and four appendices. A new article introduces the notion of connected component, which is fundamental to the qualitative leaps studied in elementary graph theory and in advanced topology; the introduction of this notion forces the recognition of the role of functors.

The appendices use examples from the text to sketch the role of adjoint functors in guiding mathematical constructions. Although these condensed appendices cannot substitute for a more detailed study of advanced topics, they will enable the student, armed with what has been learned from the text, to approach such study with greater understanding.

Buffalo, January 8, 2009

F. William Lawvere
Stephen H. Schanuel

Organisation of the book

The reader needs to be aware that this book has two very different kinds of 'chapters':

The **Articles** form the backbone of the book; they roughly correspond to the written material given to our students the first time we taught the course.

The **Sessions**, reflecting the informal classroom discussions, provide additional examples and exercises. Students who had difficulties with some of the exercises in the Articles could often solve them after the ensuing Sessions. We have tried in the Sessions to preserve the atmosphere (and even the names of the students) of that first class. The more experienced reader could gain an overview by reading only the Articles, but would miss out on many illuminating examples and perspectives.

Session 1 is introductory. Exceptionally, Session 10 is intended to give the reader a taste of more sophisticated applications; mastery of it is not essential for the rest of the book.

Each Article is further discussed and elaborated in the specific subsequent Sessions indicated below:

Article I	Sessions 2 and 3
Article II	Sessions 4 through 9
Article III	Sessions 11 through 17
Article IV	Sessions 19 through 29
Article V	Sessions 30 and 31
Article VI	Sessions 32 and 33
Article VII	Sessions 34 and 35

The **Appendices**, written in a less leisurely manner, are intended to provide a rapid summary of some of the main possible links of the basic material of the course with various more advanced developments of modern mathematics.

Acknowledgements

First Edition

This book would not have come about without the invaluable assistance of many people:

Emilio Faro, whose idea it was to include the dialogues with the students in his masterful record of the lectures, his transcriptions of which grew into the Sessions;
Danilo Lawvere, whose imaginative and efficient work played a key role in bringing this book to its current form;
our students (some of whom still make their appearance in the book), whose efforts and questions contributed to shaping it;
John Thorpe, who accepted our proposal that a foundation for discrete mathematics *and* continuous mathematics could constitute an appropriate course for beginners.

Special thanks go to Alberto Peruzzi, who provided invaluable expert criticism and much encouragement. Many helpful comments were contributed by John Bell, David Benson, Andreas Blass, Aurelio Carboni, John Corcoran, Bill Faris, Emilio Faro, Elaine Landry, Fred Linton, Saunders Mac Lane, Kazem Mahdavi, Mara Mondolfo, Koji Nakatogawa, Ivonne Pallares, Norm Severo, and Don Schack, as well as by many other friends and colleagues. We are grateful also to Cambridge University Press, in particular to Roger Astley and Maureen Storey, for all their work in producing this book.

Above all, we can never adequately acknowledge the ever-encouraging generous and graceful spirit of Fatima Fenaroli, who conceived the idea that this book should exist, and whose many creative contributions have been irreplaceable in the process of perfecting it.

Thank you all,

Buffalo, New York
2009

F. William Lawvere
Stephen H. Schanuel

Second Edition

Thanks to the readers who encouraged us to expand to this second edition, and thanks to Roger Astley and his group at Cambridge University Press for their help in bringing it about.

2009

F. William Lawvere
Stephen H. Schanuel

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Preview

SESSION 1

Galileo and multiplication of objects

1. Introduction

Our goal in this book is to explore the consequences of a new and fundamental insight about the nature of mathematics which has led to better methods for understanding and using mathematical concepts. While the insight and methods are simple, they are not as familiar as they should be; they will require some effort to master, but you will be rewarded with a clarity of understanding that will be helpful in unravelling the mathematical aspect of any subject matter.

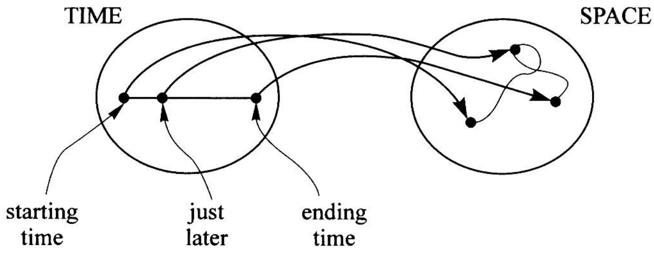
The basic notion which underlies all the others is that of a *category*, a 'mathematical universe'. There are many categories, each appropriate to a particular subject matter, and there are ways to pass from one category to another. We will begin with an informal introduction to the notion and with some examples. The ingredients will be objects, maps, and composition of maps, as we will see.

While this idea, that mathematics involves different categories and their relationships, has been implicit for centuries, it was not until 1945 that Eilenberg and Mac Lane gave *explicit* definitions of the basic notions in their ground-breaking paper 'A general theory of natural equivalences', synthesizing many decades of analysis of the workings of mathematics and the relationships of its parts.

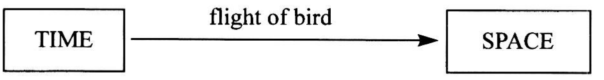
2. Galileo and the flight of a bird

Let's begin with Galileo, four centuries ago, puzzling over the problem of motion. He wished to understand the precise motion of a thrown rock, or of a water jet from a fountain. Everyone has observed the graceful parabolic arcs these follow; but the motion of a rock means more than its track. The motion involves, for each instant, the position of the rock at that instant; to record it requires a motion picture rather than a time exposure. We say the motion is a 'map' (or 'function') from time to space.

The flight of a bird as a map from time to space



Schematically:

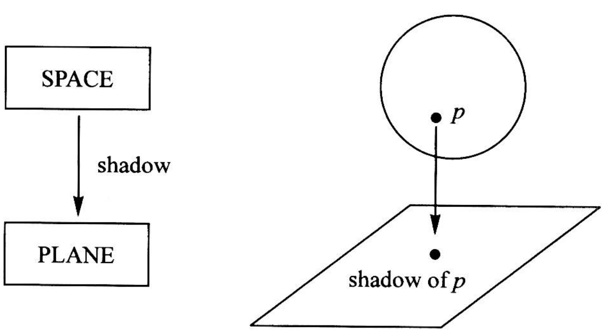


You have no doubt heard the legend; Galileo dropped a heavy weight and a light weight from the leaning tower of Pisa, surprising the onlookers when the weights hit the ground simultaneously. The study of vertical motion, of objects thrown straight up, thrown straight down, or simply dropped, seems too special to shed much light on general motion; the track of a dropped rock is straight, as any child knows. However, the motion of a dropped rock is not quite so simple; it accelerates as it falls, so that the last few feet of its fall takes less time than the first few. Why had Galileo decided to concentrate his attention on this special question of vertical motion? The answer lies in a simple equation:

$$\text{SPACE} = \text{PLANE} \times \text{LINE}$$

but it requires some explanation!

Two new maps enter the picture. Imagine the sun directly overhead, and for each point in space you'll get a shadow point on the horizontal plane:



This is one of our two maps: the 'shadow' map from space to the plane. The second map we need is best imagined by thinking of a vertical line, perhaps a pole stuck into the ground. For each point in space there is a corresponding point on the line, the one at the same level as our point in space. Let's call this map 'level':