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H Brezis, M G Crandall &
F Kappel (Editors)

Semigroups, theory and applications

VOLUME II

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H Brezis, M G Crandall &
F Kappel (Editors)

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Semigroups, theory and applications
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Preface

The Autumn Course of 1984 held at the International Centre for Theoretical Physics (ICTP) in Miramare (Trieste, Italy) during the period November 12 to December 14, 1984, was devoted to the theme "Semigroups, Theory and Applications". In accordance with the basic aims of the ICTP to promote scientific maturity of developing countries the structure of the course was the following: The program of the first three weeks consisted of basic courses in order to provide an introduction to various aspects of the area for participants with limited background. In the fourth week more advanced courses were devoted to topics of current research and the fifth week had the character of a conference on evolution problems. The course received considerable interest as is documented by the number of approximately 90 participants from developing countries.

This volume contains seven of the basic lectures presented in the first four weeks of the course. It is our opinion that this combination of lectures provide a useful introduction to the basic theory of C_o-semigroups and to various applications. A representative part of conference talks presented during the last week is contained in an other volume of this series.

Of course, it is our obligation to thank the funding agencies of the ICTP (Italian Government, UNESCO and IAEA) which made this course possible. We also immensly acknowledge support provided by the staff of the ICTP. Moreover, Prof. Abdus Salam, director of the ICTP, underlined the importance of this type of enterprises through his constant visible interest during the course. Professors L. Bertocchi, A.H. Hamende and H. Talafi contributed in various stages of the organization. We especially appreciate support by Prof. G. Vidossich during the preparation of the course.

The success of the course could not have been possible without the joint efforts of the lecturers, speakers and participants. Finally we want to express our thanks to Mrs. Bridget Buckley (Pitman) and to Dr. W. Dietl (IAEA), who by joint efforts made publication of these proceedings in the Research Notes possible. The excellent typing of the manuscript for this

volume was done by Mrs. G. Krois (Graz), who also provided efficient secretarial support concerning publication of these proceedings.

July 1986

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C BARDOS

Introduction to the equations of mathematical physics: application of the theory of semigroups to two model equations

Introduction

Contrary to the rather over ambitious original title, we will only describe two equations of mathematical physics in these notes: the heat equation and the transport equation. These two problems have been chosen, first of all, for their mathematical interest. The heat equation (often also called the diffusion equation) is the prototype of parabolic equations and the operator Δ is the most natural example of a generator of an analytic semigroup. The transport equation, in its most elementary form, is the prototype of non-linear hyperbolic problems and the operator $\frac{d}{dx}$ is the simplest existing non-trivial example of a generator of a strongly continuous group. The transport equation is involved in the construction of approximate solutions of the wave equation (asymptotic to the higher frequencies, immediately after the eikonal equation).

In both these cases, our essential objective is to show that the theory of semigroups does not merely stop with the proof of existence and uniqueness results but that it also provides the suitable framework in which we can tackle far deeper problems. It is to illustrate the idea that I have chosen to establish a trace formula for the heat operator which leads to the study of the relationship between this operator and geometry.

Similarly the greater part of the chapter on the transport equation is devoted to the spectral theory of the operator of neutron transport. Here the goal is to bring out the mathematical tools which could contribute to a better understanding of the (physical) phenomenon.

These two equations mentioned above come from "macroscopic" physics and this fact has also influenced my choice. Indeed from the beginning of this century the problems proposed by physics to mathematics came above all from quantum mechanics and then from theoretical physics. Of course these problems have contributed to the birth of extremely rich mathematical theories like Banach and Hilbert spaces or the theory of distributions. However since the

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Translated from the French version by S. Kesavan.

1950's we have witnessed a massive return to classical physics in the mathematical world. This phenomenon can be explained by giving two reasons:

1. The new problems confronted by engineers require a much more precise study of the phenomena of classical physics, as for instance, the problems arising in fluid mechanics. The interest shown now in turbulence and the Boltzmann equations, is an example of this.
2. The use of computers now permits us to qualitatively exploit several equations which, in the absence of explicit solutions, would otherwise lose much of their interest.

In concluding this introduction, I would like to thank the organizers of the Autumn College H. Brezis, M. Crandall and F. Kappel for having given me the opportunity to participate in it. I also wish to thank the I.C.T.P. and, in particular, the Director, Prof. A. Salam, for the warm hospitality and for the marvelous scientific possibilities it has put at the disposal of the participants.

The Heat Equation

The equation of heat propagation is written in the form

$$\frac{\partial u}{\partial t} - k\Delta u = 0, \qquad u(x,0) = \phi(x) \tag{1}$$

where k is the coefficient of thermal conductivity. This equation is derived, for instance, in the book of Schwartz [23, p. 227-228]. The hypothesis leading to (1) is that the rate of change of temperature depends only on the variation of the temperature gradient in space. Thus one has a diffusion phenomenon.

In the whole space $\mathbb{R}^d$ ($d = 1,2,3$ for example) one can explicitly solve (1), using the Fourier transform, as a function of the initial data $\phi(x)$. Assuming $k = 1$, we write

$$\left.\begin{aligned} &\frac{\partial \hat{u}}{\partial t}(\xi,t) + |\xi|^2\hat{u}(\xi,t) = 0 \\ &\hat{u}(\xi,0) = \hat{\phi}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i<x,\xi>}\phi(x)dx. \end{aligned}\right\} \tag{2}$$

The problem (1) is thus reduced to the solution of an ordinary differential equation (ξ now being a parameter in the equation (2)). We thus obtain

$\hat{u}(\xi,t)$ by the following formula:

$$\hat{u}(\xi,t) = e^{-t|\xi|^2}\hat{\phi}(\xi). \tag{3}$$

To obtain the function $u(x,t)$, it is now enough to apply the inverse Fourier transform which gives

$$u(x,t) = (\overline{F}\ e^{-t|\xi|^2}\ F\phi)(x). \tag{4}$$

Since we know that

$$F(f * g) = (2\pi)^{d/2}\ Ff\cdot Fg, \tag{5}$$

we deduce form (4) the formula

$$u(x,t) = \frac{1}{(2\pi)^{d/2}}\ \overline{F}(e^{-t|\xi|^2}) * \overline{F}F\phi = \frac{1}{(2\pi)^{d/2}}(F(e^{-t|\xi|^2})) * \phi. \tag{6}$$

Using the properties of the Fourier transform, we have

$$\overline{F}((2\pi)^{-d/2}e^{-|\xi|^2/2}) = (2\pi)^{-d/2}\ e^{-|x|^2/2}$$

for every $\lambda > 0$,

$$\overline{F}(g(\lambda\xi)) = \lambda^{-d/2}(\overline{F}g)(x/\lambda)$$

which finally gives us, on choosing $\lambda = \sqrt{2t}$,

$$u(x,t) = (4\pi t)^{-d/2}\int_{\mathbb{R}^d} e^{-|x-y|^2/4t}\ \phi(y)dy. \tag{7}$$

The formula (7) helps us to deduce the following properties.

(i) For each $t > 0$ and every $\phi \in L^2(\mathbb{R}^d)$ the function

$$u(x,t) = (T(t)\phi)(x) \tag{8}$$

is infinitely differentiable, even analytic with respect to the variables x and t.

(ii) For every initial data $\phi \geq 0$, even if it is of compact support, the solution $u(x,t)$ is positive for $t > 0$. This means that the initial signal propagates with infinite speed.

(iii) For each initial data $\phi \in L^{\infty}(\mathbb{R}^d)$ we have for all x,t

$$\inf_{x \in \mathbb{R}^d} \phi(x) \leq u(x,t) \leq \sup_{x \in \mathbb{R}^d} \phi(x).$$

The properties (i) and (ii) are characteristic of a class of evolution equations which are given the name "parabolic". The property (iii) is connected to the fact that the Laplacian is an elliptic second order differential operator. It is called the "maximum principle".

If we now consider the evolution of temperature not in the whole space $\mathbb{R}^d$ but in a bounded open subset Ω with boundary $\partial\Omega$, we have to prescribe boundary conditions on $\partial\Omega$. The most frequent and most natural boundary conditions are the following:

$$\text{(i)} \quad u\big|_{\partial\Omega} = 0, \qquad \text{(ii)} \quad \frac{\partial u}{\partial v}\Big|_{\partial\Omega} = 0. \tag{9}$$

The former is the Dirichlet condition (or problem). The latter is called the Neumann condition (or problem); (i) describes the situation in which by means of external (to Ω) actions, one keeps the temperature on the boundary fixed, for example $u = 0$; (ii) corresponds to a situation in which the boundary $\partial\Omega$ is perfectly nonconducting and the heat flux across $\partial\Omega$ is zero. To simplify the exposé we will restrict our attention to the Dirichlet problem.

To tackle the problem

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{in} \quad \Omega, \quad u\big|_{\partial\Omega} = 0, \quad u(\cdot,0) = \phi(\cdot) \tag{10}$$

one can use at least two methods:

(i) Functional Analysis

(ii) Introduction of a parametrix.

In fact it is often convenient, even necessary, to play with both techniques.

For example, one can introduce the space $H = L^2(\Omega)$ and in it define the (unbounded) operator A such that the problem (10) can be written in the form

$$\frac{du}{dt} + Au = 0, \quad u(0) = \phi. \tag{11}$$

Since Δ is not a bounded operator in $L^2(\Omega)$ and since we must take into account boundary conditions, we get

$$D(A) = \{u \in L^2(\Omega) \,|\, \Delta u \in L^2(\Omega),\ u|_{\partial\Omega} = 0\};\ Au = -\Delta u. \tag{12}$$

As $\partial\Omega$ is a set of measure zero in $\mathbb{R}^d$, the meaning of the relation $u|_{\partial\Omega} = 0$ is not obvious. Nevertheless one can remove this difficulty using the supplementary property that $\Delta u \in L^2(\Omega)$ (cf. Lions [14, p. 168] or Lions and Magenes [15]). In fact we have $D(A) = \{u \in H_0^1(\Omega) \,|\, \Delta u \in L^2(\Omega)\}$ which reduces the condition $u|_{\partial\Omega} = 0$ to the most usual Trace theorem in Sobolev spaces. Finally one can show (for sufficiently regular Ω),

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{13}$$

The operator A so defined is maximal monotone in the Hilbert space $H = L^2(\Omega)$. It then follows that the solution of problem (11) is given by the formula

$$u(t,x) = (T(t)\phi)(x). \tag{14}$$

In (14) $\{T(t)\}$ denotes a family of continuous linear operators in $L^2(\Omega)$ with the following properties

(i) $\|T(t)\| \leq 1$, for all $t > 0$ } (contraction

(ii) $T(t) \circ T(s) = T(t+s)$, for all $t,s > 0$ } semigroup)

(iii) $\frac{d}{dt}(T(t)\phi) = \Delta T(t)\phi$, for all $\phi \in D(\Delta)$.

Generalising the notion of the exponential of a bounded linear operator, the semigroup $T(t)$ is denoted $e^{t\Delta_D}$ (D, for Dirichlet).

We can show that this semigroup behaves essentially like the operator $T(t)$ described by the formula (8). More precisely we have:

(i) For every $t > 0$ and each $\phi \in L^2(\Omega)$, the function

$$u(x,t) = (e^{t\Delta_D}\phi)(x)$$

is infinitely differentiable for $x \in \Omega$, $t > 0$ (and is even analytic).

(ii) For every initial data $\phi \geq 0$, even if it is of compact support in Ω, the function

$$u(x,t) = (e^{t\Delta_D}\phi)(x)$$